Michael Bukatin

Partial Metrics and Fuzzy Equalities

Joint work with Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh

SumTopo 2009, Brno

Slides for this talk:

`http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html`

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Mathematics of partially defined elements.

Generalized distances: instead of $p(x,x) = 0$ axiom, value $p(x,x)$ expresses how far x is from being completely defined.

Generalized equalities: instead of $x = x$ being always true, value $=(x,x)$ expresses how well defined x is.

Example: interval numbers

Consider segments $[a, b]$ and $[c, d]$ on the real line.

Define the distance between them as

$$max(b, d) - min(a, c).$$

partial metrics (Steve Matthews)

$p(x, y) = p(y, x)$ (symmetry)

$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$

$p(x, x) \leq p(x, y)$ (small self-distance)

$p(y, y) + p(x, z) \leq p(x, y) + p(y, z)$ (strong triangularity (Steve Vickers))

Generalized metrization of various (usually non-Hausdorff) topologies and bi-topologies.

Main case for computer science so far: the Scott topology.

In the bitopological setting: dually, lower topology, and Lawson topology as join of Scott and lower.

Example: partially defined functions

Consider topological space X, set Y, and set of pairs (f, U), where U is an open subset of X and $f : U \to Y$.

The degree of equality of two functions (f, U) and (g, V) is the interior of $\{x \in U \cap V \mid f(x) = g(x)\}$.

Ω-sets

Ω-valued fuzzy equalities

D.Scott,M.Fourman,D.Higgs (1970s)

Ω – complete Heyting algebra

complete lattice, $\sqsubseteq$

for all a, b, there is greatest x, denoted as $a \to b$, such that $a \wedge x \sqsubseteq b$.

A topology is a typical complete Heyting algebra: $\sqsubseteq = \subseteq$, $\wedge = \& = \cap$, $U \to V = \mathrm{Int}(V \cup \bar{U})$.

Ω-valued fuzzy equality: $E : A \times A \to \Omega$

Axioms:

$E(a,b) = E(b,a)$

$E(a,b) \wedge E(b,c) \sqsubseteq E(a,c)$

Partial ultrametrics, $p(x,z) \leq max(p(x,y), p(y,z))$, can be viewed as fuzzy equalities.

$[0, +\infty]$ can be thought of as the Scott topology on positive reals.

$\sqsubseteq = \subseteq = \geq$ (order is reversed)

partial ultrametrics

$p(x, y) = p(y, x)$ (symmetry)

$p(x, x) = p(x, y) = p(y, y) \Rightarrow x = y$

$p(x, z) \leq max(p(x, y), p(y, z))$

Partial ultrametrics are partial metrics.

Fourman and Scott also introduced a mechanism of *singletons*, which was used to define the notion of *complete Ω-set* and to establish that complete Ω-sets and sheaves over complete Heyting algebra Ω are essentially the same thing.

[see Fourman M. P., D. S. Scott, *Sheaves and Logic*, in "Applications of Sheaf Theory to Algebra, Analysis, and Topology," Lecture Notes in Mathematics, **753**, Springer, 1979, pp. 302–401.]

Quantale-valued partial metrics

R.Kopperman, S.Matthews, H.Pajoohesh (2004)

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a + b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a+c \geq b\}$. Certain additional conditions are imposed.

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ are

- $p(x, x) \leq p(x, y)$
- $p(x, y) = p(y, x)$
- $p(x, z) \leq p(x, y) + (p(y, z) \dot{-} p(y, y))$

Quantale-valued sets

Quantale-valued fuzzy equalities

Ulrich Hoehle (early 1990s)

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

An M-valued set is a set X equipped with a map $E : X \times X \to M$ (**fuzzy equality**) subject to the axioms

- $E(x, y) \sqsubseteq E(x, x)$

- $E(x, y) = E(y, x)$

- $E(x, y) * (E(y, y) \Rightarrow E(y, z)) \sqsubseteq E(x, z)$

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We noticed the equivalence between partial metrics and fuzzy equalities in 2006:

http://www.cs.brandeis.edu/~bukatin/distances_and_equalities.html

Why do we need quantale generalizations?

partial metrics – generalized metrization of large topologies

fuzzy equalities – non-idempotent conjunctions such as Lukasiewicz conjunction:

$x \& y = max(0, x + y - 1)$.

weak semi-cancellativity:

$a \geq b \Rightarrow a = (a \dot{-} b) + b.$

Hoehle imposes its equivalent, and we should impose it also in the theory of partial metrics.

Then we can rewrite the strong triangularity as

$p(y,y) + p(x,z) \leq p(x,y) + p(y,z)$

$E(x,y) * E(y,z) \sqsubseteq E(x,z) * E(y,y)$

connections with linear logic

$$p(y,y)+p(x,z) \leq p(x,y) + p(y,z)$$

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

$$q(x,y) = p(x,y) - p(x,x)$$

$$d(x,y) = q(x,y) + q(y,x) = 2p(x,y) - p(x,x) - p(y,y)$$

Metric-entropy pairs

Dan Simovici

Metric-Entropy Pairs on Lattices, Journal of Universal Computer Science (Springer-Verlag), vol. **13**, no.11, 2007, pp. 1767-1778

http://www.cs.umb.edu/∼dsim/papersps/de.pdf

Definition 1, formula (1): Consider a lattice L, and functions $d : L \times L \to [0, +\infty)$ and $\eta : L \to [0, +\infty)$. The pair (d, η) is a $\wedge$-pair if $d(x, y) = 2\eta(x \wedge y) - \eta(x) - \eta(y)$.

Theorem 4, formula (3): Given a $\wedge$-pair (d, η), axiom $d(x, y) \le d(x, z) + d(z, y)$ holds if and only if $\eta(z) + \eta(x \wedge y) \le \eta(x \wedge z) + \eta(y \wedge z)$ for all x, y, z.

Consider $p(x, y) = \eta(x \wedge y)$.

Then $p(x, x) = w(x) = \eta(x)$.

How can we use the knowledge, that we see a common pattern here, which reappears in different contexts?

1) We can transfer methods, results, and intuition between these contexts (there are multiple examples of this).

2) We might be able to create a more abstract theory covering these patterns, perhaps of a categorical flavor. (In 1973 Lawvere established a connection between quasi-metrics, partial orders, and enriched categories; here we are likely to have a similar connection between partial metrics, generalized equalities, and sheaves. The manuscript by Higgs contains a construction of fuzzy equalities into Grothendieck sites.)

3) Better math on the spaces of programs / spaces of computational processes?

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