On a Smyth Conjecture*

Michael A. Bukatin[1] and Svetlana Yu. Shorina[2]

[1] Department of Computer Science, Brandeis University, Waltham, MA 02254, USA;

bukatin@cs.brandeis.edu; *http://www.cs.brandeis.edu/~bukatin/papers.html*

[2] Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia;

sveta@cpm.ru

June 19, 2000

Abstract. The set $I_x = \{y \in A \mid \{x, y\}$ is unbounded$\}$ is an observable continuous representation of negative information about $x \in A$ for a weakly Hausdorff continuous dcpo A with the Scott topology. When A is not weakly Hausdorff the largest continuous approximation of I_x is represented by $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$, and the largest observable continuous representation of I_x is represented by $J'_x = \mathrm{Int}(J_x)$.

Mike Smyth conjectured, that J or J' is closely related to the least symmetric closed tolerance on A. In this paper we establish that, indeed, $\{\langle x, y\rangle \mid y \in J'_x\}$ is the complement of this tolerance.

We also establish a relationship between this tolerance and lower bounds of relaxed metrics on A.

1 Introduction

Recently Mike Smyth [10] and Julian Webster [13] advanced the approach in which tolerance is considered not as an alternative to standard topology, but as a complementary structure to topology. In particular, it seems to be fruitful to equip Scott domains with tolerances.

The present paper represents a small contribution to this emerging theory.

1.1 Negative Information for the Weakly Hausdorff Case

Consider a continuous, bounded complete dcpo A (continuous Scott domain). Then one of the natural ways to represent negative information about $x \in A$ is via set $I_x = \{y \in A \mid \{x, y\}$ is unbounded$\}$. It turns out that such negative information is observable ($\forall x \in A.\, I_x$ is Scott open) and continuous ($\forall B \subseteq A.\, B$ is directed $\Rightarrow I_{\sqcup B} = \bigcup_{b \in B} I_b$).

Consider a generalization to the case, where we no longer require A to be bounded complete. It turns out that negative information I_x is continuous if and only if it is observable [3].

Escardo [6] (see also [10]) defined a topological space A to be *weakly Hausdorff*, if its consistency relation is closed. The consistency relation is given by formula $x \uparrow y = \{\langle x, y\rangle \mid \exists z \in A.\, x \sqsubseteq z \& y \sqsubseteq z\}$, where $\sqsubseteq$ is the specialization order of A.

A continuous dcpo with Scott topology is weakly Hausdorff if and only if negative information is observable (or continuous), due to the fact that the relation $\{\langle x, y\rangle \mid y \in I_x\}$ is exactly the complement of the consistency relation.

1.2 The Least Closed Tolerance

Smyth [10] defines a tolerance as a reflexive symmetric relation following Poincare and Zeeman. He defines a topological tolerance space as a topological space equipped with a tolerance relation closed in the product topology.

*Supported by **Applied Continuity in Computations Project**.

For a weakly Hausdorff space, $\uparrow$ is the least closed tolerance.

1.3 Negative Information for Continuous Dcpo's

In [3] we proved, that the largest continuous approximation of I_x is given by $J_x = \{y \in A \mid x \in \mathrm{Int}(I_y)\}$ and that the largest observable continuous approximation of I_x is given by $J'_x = \mathrm{Int}(J_x)$, when A is a continuous dcpo.

Note, that the continuous dcpo A is weakly Hausdorff iff $I = J$ iff $I = J = J'$.

By representing negative information about $x \in A$ via J_x or J'_x, we were able to extend our approach for obtaining Scott continuous relaxed metrics [1] from bi-continuous valuations [7, 2] beyond bounded complete domains [3].

More specifically, in [2] we introduced CC-valuations, that is continuous valuations μ, which are co-continuous (if $\{U_i, i \in I\}$ is a filtered system of open sets, then $\mu(\mathrm{Int}(\bigcap_{i\in I} U_i)) = \inf_{i\in I} \mu(U_i)$), strongly non-degenerate (if $U \subset V$ [1] are open sets, then $\mu(U) < \mu(V)$), and normalized ($\mu(A) = 1$). Klaus Keimel [7] showed the existence of a CC-valuation for any continuous dcpo with a countable basis. In [3] we suggested that given a CC-valuation μ on a continuous dcpo A, a relaxed distance between $x, y \in A$ should be defined as an interval number $\langle l(x,y), u(x,y)\rangle$, where $l(x,y) = \mu(C_x \cap J'_y) + \mu(C_y \cap J'_x)$, $u(x,y) = 1 - \mu(C_x \cap C_y) - \mu(J'_x \cap J'_y)$, $C_x = \{y \in A \mid y \sqsubseteq x\}$. We established that such relaxed metrics are Scott continuous and yield the Scott topology on A.

Our approach worked especially well, when a continuous dcpo A obeyed the *Lawson condition*, that the relative Scott and Lawson topologies on $Total(A)$ coincide [8]. In this case, the induced relaxed metrics on A became ordinary metrics when restricted to $Total(A)$. In [3] we established that for a continuous dcpo A the Lawson condition is equivalent to the formula $\forall x \in Total(A).\, I_x = J_x$. Since this is a weakening of $I = J$, which is equivalent to A being weakly Hausdorff, we call the spaces satisfying the Lawson condition *very weakly Hausdorff*.

1.4 A Smyth Conjecture

Mike Smyth suggested that either J or J' should be closely related to the least closed tolerance on continuous dcpo's. Initially we turned this conjecture aside, because the tolerances in question are symmetric, and the relation $\{\langle x,y\rangle \mid y \in J_x\}$ is, in general, not symmetric.

However, recently we returned to this conjecture and, to our surprise, discovered that the relation $x \not\uparrow y = \{\langle x,y\rangle \mid y \in J'_x\}$ is symmetric and is, indeed, the complement of the least closed tolerance on any continuous dcpo.

1.5 Structure of the Paper

In Section 2 we recall basic definitions of domain theory. In Section 3 we review our earlier results on J and J'.

In Section 4 we talk about tolerances and prove the central result of this paper.

In Section 5 we establish that for the relaxed metrics defined above, $x \not\uparrow y$ if and only if $l(x,y) \neq 0$. We also build a continuous family of tolerances, $\{\langle x,y\rangle \mid l(x,y) \leq \epsilon\}$, parametrized by ϵ.

In Conclusion we focus on some of the potential uses of tolerances.

2 Basic Notions of Domain Theory

The following basic notions and facts can be found in [4].

[1] We use $U \subset V$ as an equivalent of $U \subseteq V \& U \neq V$.

Recall that a non-empty partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. A poset, $(A, \sqsubseteq)$, is a *dcpo* if it has the least element, $\perp$, and for any directed $S \subseteq A$, the least upper bound $\sqcup S$ of S exists in A. A set $U \subseteq A$ is *Alexandrov open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$. A set $U \subseteq A$ is *Scott open* if it is Alexandrov open and for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$. The Scott open subsets of a dcpo form the *Scott topology*.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. $f : A \to B$ is (Scott) continuous iff it is monotonic ($x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$) and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

We define continuous dcpo's and Scott domains in the spirit of [4]. Consider a dcpo $(A, \sqsubseteq)$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$.

Consider a set $K \subseteq A$. We say that the dcpo A is a *continuous dcpo* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements. Notice that $\perp_A \in K$.

We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B$ exists. A continuous, bounded complete dcpo is called a *continuous Scott domain.*

We will also need the following result from [2].

Lemma 2.1 (Border Lemma) *Consider a continuous dcpo A and an Alexandrov open set $B \subseteq A$. Then its interior in the Scott topology,* $\mathrm{Int}(B) = \{y \in A \,|\, \exists x \in B.\ x \ll y\}$. *Correspondingly, the border of B in the Scott topology,* $B \setminus \mathrm{Int}(B) = \{y \in B \,|\, \neg(\exists x \in B.\ x \ll y)\}$

All interiors in this paper are taken in the Scott topology.

3 Our Earlier Results on Negative Information

This section reviews some of our earlier results from [3].

3.1 Negation Duality and Problems with I_x

It is convenient to use *negation duality*, $x \in I_y \Leftrightarrow y \in I_x$, when analyzing the behavior of I_x. Specifically, one can consider directed sets $B \subseteq A$, and by considering $x = \sqcup B$ and $x = b \in B$, obtain the following Lemma:

Lemma 3.1 *For any dcpo A,* $(\forall y \in A.\, I_y$ *is Scott open*$) \Leftrightarrow (\forall B \subseteq A.\, B$ *is directed* $\Rightarrow I_{\sqcup B} = \bigcup_{b \in B} I_b)$.

Proof.

$\Rightarrow$. A non-trivial part is to show $I_{\sqcup B} \subseteq \bigcup_{b \in B} I_b$. Consider $y \in I_{\sqcup B}$. Hence $\sqcup B \in I_y$. Since I_y is Scott open, there is $b \in B$, such that $b \in I_y$. Hence $y \in I_b$.

$\Leftarrow$. Obviously, I_y is Alexandrov open. Consider a directed set B, such that $\sqcup B \in I_y$. Hence $y \in I_{\sqcup B}$, hence $\exists b \in B.\, y \in I_b$, hence $b \in I_y$.

□

One can informally restate this Lemma by saying that all I_y are (Scott) observable [9], if and only if $I : y \mapsto I_y$ is a Scott continuous function $A \to \mathcal{P}(A)$.

For some continuous dcpo's certain I_y's are not Scott open and the corresponding equality $I_{\sqcup B} = \bigcup_{b \in B} I_b$ does not hold for some directed sets B (see Example below).

3.2 Example

Let us start with an example. We define an algebraic countable dcpo E, as the following subset of the powerset of $\mathbf{Z}$, ordered by subset inclusion. $E = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \ldots, \{1,2,3,\ldots\}, \{0\}, \{1,0,-1\}, \{1,2,0,-2\}, \{1,2,3,0,-3\}, \ldots\}$. For convenience, we introduce a unique letter denotation for each of the elements of E: $\perp_E = \emptyset$, $e_1 = \{1\}$, $e_2 = \{1,2\}$, $e_3 = \{1,2,3\}, \ldots, e_\infty = \{1,2,3,\ldots\}$, $0_E = \{0\}$, $f_1 = \{1,0,-1\}$, $f_2 = \{1,2,0,-2\}$, $f_3 = \{1,2,3,0,-3\}, \ldots$ We will use this notation throughout the paper. Observe, that all elements, except e_∞, are compact, that elements $e_\infty, f_1, f_2, \ldots$ are total, and that $e_i \sqsubseteq f_j$ iff $i \leq j$.

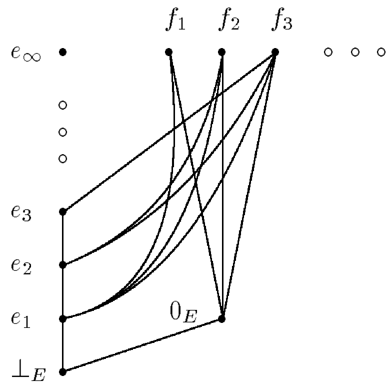

Figure 1: Domain E

Now we will see how *negation duality* works in this example. In our previous terminology, $x = e_\infty$, $y = 0_E$. The role of a directed set B is played by the increasing sequence, $e_1 \sqsubseteq e_2 \sqsubseteq e_3 \sqsubseteq \ldots$. Notice that $e_\infty = \sqcup B$.

Note also that $I_{0_E} = \{e_\infty\}$ and $0_E \in I_{e_\infty}$. We see, that I_{0_E} is not Scott open (although I_{e_∞} is Scott open), and dually, we obtain that $0_E \notin \bigcup_{b \in B} I_b$ (due to the observation, that $I_{e_1} = \emptyset, I_{e_2} = \{f_1\}, I_{e_3} = \{f_1, f_2\}, \ldots$), thus breaking $I_{\sqcup B} = \bigcup_{b \in B} I_b$.

3.3 Solution

Let us rewrite the negation duality as $I_x = \{y \mid x \in I_y\}$. What works, somewhat surprisingly, is taking a subset of I_x via taking the interior inside the right-hand side of this expression: $J_x = \{y \mid x \in \mathrm{Int}(I_y)\}$.

Lemma 3.2 *If B is a directed set, $J_{\sqcup B} = \bigcup_{b \in B} J_b$.*

In the next Subsection, we explain this result in terms of Stone duality. In our example domain E, J_{e_∞} does not include 0_E, unlike I_{e_∞}, yielding $J_{e_\infty} = \bigcup_i J_{e_i}$.

In general, J_x is Alexandrov open, but does not have to be Scott open. E.g., in our example domain E, we have that $J_{0_E} = I_{0_E} = \{e_\infty\}$, and thus J_{0_E} is not Scott open. We can replace J by $J'_x = \mathrm{Int}(J_x)$ due to the following Lemma.

Lemma 3.3 *If for arbitrary Alexandrov open sets $J, J_m, m \in M$, the equality $J = \bigcup_{m \in M} J_m$ holds, then $\mathrm{Int}(J) = \bigcup_{m \in M} \mathrm{Int}(J_m)$ holds as well.*

3.4 Stone Duality

Lemma 3.2 can be generalized as follows.

For the purpose of this Subsection only, assume that there is a continuous dcpo A and a set D, and that we are given a map $U : D \to \mathcal{O}(A)$, where $\mathcal{O}(A)$ is the frame of Scott open sets of the domain A.

Now generalize the construction of J_x by considering the map $J : A \to \mathcal{P}(D)$, where $\mathcal{P}(D)$ is the powerset of D ordered by set-theoretic inclusion and equipped with the Scott topology. Define J by the formula: $J : x \mapsto \{y \in D \mid x \in U(y)\}$. Then one can prove that for any directed set B, $J_{\sqcup B} = \bigcup_{b \in B} J_b$, implying that J is a Scott continuous function from A to $\mathcal{P}(D)$.

Let us analyze this situation in the spirit of *Stone duality* [5, 11], which is a contravariant equivalence between categories of spatial frames (of open sets) and sober topological spaces.

Observe that, if one applies J^{-1} to a subbasic open set $\uparrow\{y\}$, one obtains $J^{-1}(\uparrow\{y\}) = \{x \mid J(x) \in \Uparrow\{y\}\} = \{x \mid y \in J(x)\} = \{x \mid x \in U(y)\} = U(y)$.

Thus the map U can be thought of as defined on the generators $\Uparrow\{y\}$ of the free frame of all Scott open sets on $\mathcal{P}(D)$ and giving rise to the frame homomorphism $u : \mathcal{O}(\mathcal{P}(D)) \to \mathcal{O}(A)$ (of course, $u = J^{-1}$, e.g. for basic open sets, $u(\Uparrow\{y_1, \ldots, y_n\}) = U(y_1) \cap \ldots \cap U(y_n)$, and the similar thing goes for unions of basic sets).

Now, since we are dealing with sober spaces (Scott topologies of continuous dcpo's are sober [5]), Stone duality means not only that $u = J^{-1}$ can be obtained from the continuous function J, but also that the continuous function J can be restored from the frame morphism u. And this is the essence of our definition of J, when we think about U as defined on the generators of the frame $\mathcal{O}(\mathcal{P}(D))$.

3.5 J_x is the Largest Continuous Approximation of I_x

Both I_x and J_x can be considered as functions from A to the powerset of A, $\mathcal{P}(A)$. However, in general, only J_x is Scott continuous. The following theorem shows that, in some sense, J_x is the best we can do.

Theorem 3.1 *If $f : A \to \mathcal{P}(A)$ is a Scott continuous function and $\forall x \in A.\, f(x) \subseteq I_x$, then $\forall x \in A.\, f(x) \subseteq J_x$.*

A similar statement holds for $\mathrm{Int}(I_x)$ and $J'_x = \mathrm{Int}(J_x)$, understood as functions from A to the dual domain of open sets of A.

4 Tolerances and Negative Information

4.1 Tolerances, Distinguishability, and Observability

Smyth [10] requires that a tolerance relation is closed in the product topology. Here are informal reasons for this.

The typical meaning of two points being in the relation of tolerance, $x \sim y$, is that x cannot be distinguished from y, i.e. there is no way to establish, that x and y differ.

The natural way to interpret the statement, that x and y can be distinguished, is to give some "effective" procedure for making such a distinction. Thus, the property of being distinguishable is observable [9]. Correspondingly, the property $x \sim y$ is refutable, hence $\sim$ should be closed.

The fact that the least closed tolerance for a weakly Hausdorff continuous dcpo is $\uparrow$ also is quite natural in this framework. Indeed, domain elements are thought of as being only partially known and dynamically increasing in the course of their lives. Hence the fact that $x \uparrow y$, that is $\exists z.\, x \sqsubseteq z, y \sqsubseteq z$, precisely means that x and y may approximate the same "genuine" element z, hence we cannot distinguish between them. Since $\uparrow$ is closed in the weakly Hausdorff case, its complement is open, hence observable. That means that when $x \uparrow y$ does not hold, there is some "finite" way to distinguish between x and y.

4.2 J' and the Least Closed Tolerance (a Smyth Conjecture)

Consider a continuous dcpo A. In this Subsection $x, y, v, w \in A$. Recall that we defined $x \not\sim y = \{\langle x, y\rangle \mid x \in J'_y\}$.

Lemma 4.1 $x \not\sim y = \{\langle x, y\rangle \mid \exists \langle v, w\rangle.\, v \ll x, w \ll y, \{v, w\}$ *is unbounded*$\}$.

Proof. Using the Border Lemma we get $x \in \mathrm{Int}(J_y)$ iff $\exists v \in J_y.\, v \ll x$. By the definition of J_y, $v \in J_y$ iff $y \in \mathrm{Int}(I_v)$ i.e. $\exists w \in I_v.\, w \ll y$. Finally recall that $w \in I_v$ iff $\{v, w\}$ is unbounded .

□

It is an immediate corollary that $\not\sim$ is symmetric.

Let us show that $\not\sim$ is open in the product topology. If we fix a pair $\langle v, w\rangle$ the set $\not\sim_{\langle v,w\rangle} = \{x \mid v \ll x\} \times \{y \mid w \ll y\}$ is open, and our $\not\sim$ is the union of all such sets for all unbounded pairs $\langle v, w\rangle$.

Theorem 4.1 *The relation* $\not\sim$ *is the complement of the least closed tolerance.*

Proof. Consider an open set $W \subseteq X \times X$, such that $\not\sim \subset W$. Consider a pair $\langle x, y\rangle \in W \setminus \not\sim$. Since W is open, we can choose two open sets $U \subseteq X$ and $V \subseteq X$, such that $\langle x, y\rangle \in U \times V \subseteq W$. Consider a pair $\langle p, r\rangle, p \in U, r \in V. p \ll x, r \ll y$. The pair $\langle p, r\rangle$ is bounded, otherwise $\langle x, y\rangle \in \not\sim$, so we can take z such that $p \sqsubseteq z$, $r \sqsubseteq z$, therefore $z \in U$, $z \in V$, so $\langle z, z\rangle \in U \times V$ and $\langle z, z\rangle \in W$. So the complement of W is not a tolerance, because it is not reflexive.

□

4.3 Examples

In our example domain E, the pair $\langle e_\infty, 0_E\rangle$ is unbounded, but belongs to the least closed tolerance, since these elements cannot be distinguished by looking at the approximation pairs, $\langle e_1, 0_E\rangle$, $\langle e_2, 0_E\rangle, \ldots$.

Moreover, by adding to domain E elements $g_2 = \{0, 2\}, g_3 = \{0, 2, 3\}, \ldots, g_n = \{0, 2, 3, \ldots, n\}$, ... and $g_\infty = \{0, 2, 3, 4, \ldots\}$, we obtain a domain, where two different maximal elements, e_∞ and g_∞, cannot be distinguished via finite observations, because all their approximations are bounded. Such situation when $\exists x, y \in Total(A).\, x \neq y \& x \sim y$, where $\sim$ is the least closed tolerance, cannot occur in the presence of the Lawson condition, because the Lawson condition is equivalent to $\forall x \in Total(A).\, I_x = J'_x$.

5 Tolerances and Lower Bounds of Relaxed Metrics

We are going to prove the following statement.

Theorem 5.1 $x \not\sim y \Leftrightarrow l(x, y) \neq 0$.

Proof. $\Rightarrow$. Recall from Subsection 1.3 that $l(x, y) = \mu(C_x \cap J'_y) + \mu(C_y \cap J'_x)$. $x \not\sim y$ implies $x \in C_x \cap J'_y$, which implies that $C_x \cap J'_y \neq \emptyset$. Hence $J'_y \setminus C_x \subset J'_y$, hence $\mu(C_x \cap J'_y) = \mu(J'_y) - \mu(J'_y \setminus C_x) > 0$ due to the strong non-degeneracy of μ. Hence $l(x, y) > 0$.

$\Leftarrow$. $l(x, y) > 0$ means $C_x \cap J'_y \neq \emptyset$ or $C_y \cap J'_x \neq \emptyset$. It is enough to consider $C_x \cap J'_y \neq \emptyset$. Since x is the largest element of C_x, we obtain $x \in J'_y$, hence $x \not\sim y$. □

Only upper bounds of relaxed metrics participate in the definition of the relaxed metric topology. Hence lower bounds are usually considered as only playing an auxiliary role in the computation of upper bounds. Here we see an example of a quite different application of lower bounds.

5.1 A Continuous Family of Tolerances

A set, $\{\langle x, y\rangle \mid l(x,y) \leq \epsilon\}$, also forms a tolerance. Indeed, this is a symmetric, reflexive relation. To see that it is closed, consider a Scott continuous function $l : A \times A \to R^+$, where R^+ is a segment $[0,1]$ with the usual ordering and the induced Scott topology, and observe that the set in question is the inverse image of a Scott closed set $[0,\epsilon] \subseteq R^+$ under l^{-1}.

The resulting family of tolerances parametrized by ϵ is Scott continuous in the following sense. The dual domain for R^+ is domain $R^-_\top$. Here R^- is the same segment of numbers, but with inverse ordering ($x \sqsubseteq y \Leftrightarrow x \geq y$), an element $r \in R^-$ corresponds to the open set $(r,1] \subseteq R^+$, the element $\top \in R^-_\top$ corresponds to the open set $[0,1] = R^+$, and domains R^- and $R^-_\top$ are equipped with the induced Scott topology.

The function $l^{-1} : R^-_\top \to \mathcal{O}(A \times A)$ is Scott continuous, and so is its restriction on R^-. Then $l^{-1}(\epsilon)$ is the complement of the tolerance in question, and we can think about this complement as representing this tolerance in the dual domain $\mathcal{O}(A \times A)$.

6 Conclusion

It seems that tolerances will play an increasingly important role in domain theory. One particularly promising direction of development is to use tolerances and especially their asymmetric generalizations instead of transitivity of logical inference to formally express and study the ideas of A.S.Esenin-Vol'pin and P.Vopenka, that large numbers should be considered infinite, and long proofs and computations should be considered meaningless [12].

Acknowledgements

We would like to thank Joshua Scott, Mike Smyth, Steve Vickers, and Julian Webster for helpful discussions.

References

[1] Bukatin M.A., Scott J.S. Towards computing distances between programs via Scott domains. In S. Adian, A. Nerode, eds., Logical Foundations of Computer Science, *Lecture Notes in Computer Science*, **1234**, 33–43, Springer, 1997.

[2] Bukatin M.A., Shorina S.Yu. Partial metrics and co-continuous valuations. In M. Nivat, ed., Foundations of Software Science and Computation Structures, *Lecture Notes in Computer Science*, **1378**, 125–139, Springer, 1998.

[3] Bukatin M.A., Shorina S.Yu. *Relaxed Metrics, Maximal Points, and Negative Information*, Submitted to MFPS 14 Proceedings, 1998.

Available via URL

http://www.cs.brandeis.edu/~bukatin/neginfo.{dvi,ps}

[4] Hoofman R. Continuous information systems. *Information and Computation*, **105** (1993), 42–71.

[5] Johnstone P.T. *Stone Spaces*, Cambridge University Press, 1982.

[6] Escardo M.H. *PCF Extended with Real Numbers: A Domain-Theoretic Approach to Higher-Order Exact Real Number Computation.* PhD Thesis, University of London, Imperial College, 1996.

Available via URL

http://www.dcs.ed.ac.uk/lfcsreps/EXPORT/97/ECS-LFCS-97-374/index.html

[7] Keimel K. Bi-continuous Valuations. In A. Edalat et al., eds., Third Workshop on Computation and Approximation, *Electronic Notes in Theoretical Computer Science*, **13**, Elsevier, 1998.

[8] Lawson J. Spaces of Maximal Points. *Mathematical Structures in Computer Science*, **7** (1997), 543–555.

[9] Smyth M.B. Topology. In S. Abramsky et al., eds., *Handbook of Logic in Computer Science*, Vol.1, 641–761, Claredon Press, Oxford, 1992.

[10] Smyth M.B. Topology and Tolerance. In S. Brookes, M. Mislove, eds., Mathematical Foundations of Programming Semantics, *Electronic Notes in Theoretical Computer Science*, **6**, Elsevier, 1997.

[11] Vickers S. *Topology via Logic*, Cambridge University Press, 1989.

[12] Vopenka P. Mathematics in the Alternative Set Theory, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1979.

[13] Webster J. *Topology and Measure Theory in the Digital Setting: on the Approximation of Spaces by Inverse Sequences of Graphs.* PhD Thesis, University of London, Imperial College, 1997.

Available via URL

http://theory.doc.ic.ac.uk/people/Webster/tesi.ps