Towards Computing Distances between Programs via Domains: a Symmetric Continuous Generalized Metric for Scott Topology on Continuous Scott Domains with Countable Bases*

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Abstract

This paper presents a description of Scott topology on continuous Scott domains via continuous generalized distances. The continuity of these distances makes it possible to compute them in many cases. In particular, applications of this construction to semantic domains allow to compute meaningful distances between programs.

The paper starts with a very simple variation of the Kopperman-Flagg generalized asymmetric distance for the Scott topology on a Scott domain described in [3]. This variation works for any continuous lattice with a countable basis and makes all distances non-negative real numbers. Unfortunately, the resulting distance is not continuous, which prevents its effective computation in many cases.

A different version of generalized metric is then developed. The values are interpreted as upper estimates of real numbers. This metric is Scott continuous. It is symmetric, yet surprisingly still yields the asymmetric Scott topology. The classical metric property sacrificed is the requirement that the distance from an element to itself be zero. The intuition behind our construction is discussed.

This construction is then generalized to continuous Scott domains with countable bases. The resulting distances are interval numbers. All properties described in the previous paragraph hold. When a domain is equipped with a regular countable basis, the restriction of the generalized metric on the set of total elements yields an ordinary metric. This metric on total elements coincides, somewhat surprisingly, with the restriction of our version of the symmetrized Kopperman-Flagg distance on the set of total elements. In particular, the set of compact elements forms a regular basis for any algebraic Scott domain, so the results above hold for all ω-algebraic Scott domains.

Recently it was found that in both the continuous cases upper estimates form Matthews partial metrics [5] and the construction is closely related to the one of O'Neill [6]. This paper describes a wider class of domains than [6] in a more intuitive way from the computational point of view.

1 Introduction

The initial point for this research is our desire to find out how to compute meaningful distances between programs. By this we mean the distances related to differences in computational behavior

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of programs rather than differences in their texts. One possible route to this goal seems to go via metric structures on domains for denotational semantics. We will use [2] as the main reference for domain theory, since its setting is the closest to ours; see also references therein.

The typical setting in denotational semantics is as follows. We have a syntactic domain of programs, P, a semantic domain of meanings, S, and a continuous semantic function, $[\![\,]\!] : P \to S$.

Now assume that we have a reasonable domain of distances, D, and a continuous metric function, $\rho : S \times S \to D$. It would be reasonable to say that ρ reflects computational properties of S, if ρ yields the Scott topology on S.

Then the composition of ρ and $[\![\,]\!]$ would yield a computationally meaningful distance between programs: $< p_1, p_2 > \mapsto \rho([\![p_1]\!], [\![p_2]\!])$.

We start with the attempt to adopt the well known Kopperman-Flagg generalized distance [3] for this purpose. Initially it seems, that the only obstacle would be the fact that under this approach distances are complicated sets rather than real numbers. We want real numbers not only due to the desire to be as conventional as possible, but also because they can be expressed via several characters with high precision, whereas the textual representation of sets is likely to be very complicated, if possible at all. We keep in mind the desire to be able to actually write a program computing distances between programs.

It turns out, that it is indeed not difficult to modify the construction from [3] for Scott domains, so that the distances are real numbers and the Scott topology still results (Section 3).

Unfortunately, we discover that this distance is not continuous, and it seems unlikely, that any small modification would fix that. Connections between continuity and computability are well known, and we discuss them in further details in the Subsection 3.3. Here it is sufficient to say that the lack of continuity prevents the possibility of having a general method computing such distances.

We managed to find a different generalized metric on domains, which is continuous and can be computed, at least in a certain sense. This distance also yields the Scott topology. Unlike the Kopperman-Flagg distance our distance is symmetric. One has to sacrifice some properties of ordinary metrics in order to describe non-Hausdorff topologies. We sacrifice the property that the distance from a point to itself is zero.

Our distances are interval numbers. However, when a domain has a top element, our current approach makes all lower estimates zeros. Then only upper estimates are meaningful, and we can pretend that our distances are ordinary real numbers. We start with this case in Section 4. The general case is considered in Section 5.

The distinctive property of our metric is that the distance from a point to itself is usually not zero. Lately, due to the helpful advice of Bob Flagg, we found out that in both cases upper estimates form Matthews partial metrics [5, 4]. Our methods also turned out to be similar to the methods developed by O'Neill [6], although we managed to describe a wider class of domains and avoided the introduction of negative distances.

We elaborate on the intuition behind this property and on relationships between our work and [5, 4, 6] in the respective discussion Subsections 4.3 and 5.5. However, we would like to present a simple example here. This example illustrates the intuition involved extremely well; however, we should caution that this example is not a partial case of our current construction; also the upper estimate is not a partial metric, unlike Example 3.2 of [5].

Consider interval numbers $x = [a, b]$ and $y = [c, d]$. Think about x and y as bona fide real numbers, but known only with certain precision. Namely, think that it is known, that $a \le x \le b$ and $c \le y \le d$. So what is known about the distance between x and y? Consider the set S of real numbers, $S = \{|x' - y'| \mid a \le x' \le b, c \le y' \le d\}$. The distance between x and y is greater or equal than $\inf S$, and is smaller or equal than $\sup S$. Thus, we should say, that the distance between interval numbers x and y is the interval number $[\inf S, \sup S]$. In particular, if $x = [a, b]$ and $y = [a, b]$, the distance between them is $[0, b - a]$. The intuition here is that, although the information known about x and y is the same, they may be different numbers — each of them can be located within the interval $[a, b]$ quite independently of another.

Observe here that when one deals with precise interval numbers, precise distances result. It is easy to check that the distance between $[a, a]$ and $[b, b]$ is $[|a-b|, |a-b|]$.

We discuss how the restriction of our interval generalized metrics onto the sets of total elements yields ordinary metrics in Subsection 5.4. Directions of further research are discussed in the Conclusion. Section 2 contains basic definitions and should be used as a reference.

2 Basic notions

2.1 Continuous Scott domains with bases

We introduce basics of the theory of algebraic and continuous domains in the spirit of [2].

Consider a bounded complete dcpo A and denote its partial order as $\sqsubseteq$. We say that $x \ll y$ (x is *way below* y) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S$ implies the existence of $s \in S$, such that $x \sqsubseteq s$.

Consider a set $K \subseteq A$. Assume that A has the smallest element, $\bot$, and that $\bot \in K$. We say that A is a *continuous Scott domain* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements. If a continuous Scott domain A has a top element, $\top$, A is called a *continuous lattice*.

A set $B \subseteq A$ is *Scott open* if it is upwardly closed — $x \in B, x \sqsubseteq y$ implies $y \in B$, — and for any $x \in B$, there is $k \in K$, such that $k \in B$ and $k \ll x$. The Scott open sets form a topology which is called the *Scott topology*.

An element $c \in A$ is called *compact*, if $c \ll c$. Denote the set of all compact elements of A as C. If for any $a \in A$, the set $C_a = \{c \in C \mid c \sqsubseteq a\}$ is directed and $a = \sqcup C_a$, the bounded complete dcpo A is called an *algebraic Scott domain*. If A also has a top element, it is called an *algebraic lattice*.

An algebraic Scott domain A with a set of compact elements C is a continuous Scott domain with basis C. This is the most important subclass of continuous Scott domains.

We only consider continuous Scott domains with countable bases in this paper. One can always take the entire continuous Scott domain as a basis obtaining more classical version of the definition of a continuous Scott domain, but this leads to the loss of countability.

Below in the paper, A always denotes a continuous Scott domain, and K a countable basis.

For any $a, b \in A$, there is $k \in K$, such that $k \in K_a$, $k \notin K_b$ if and only if $a \not\sqsubseteq b$.

Given domains A and A' with bases, respectively, K and K', we say that the function $f : A \to A'$ is continuous if it is monotonic — $x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_{A'} f(y)$ — and preserves the least upper bounds of directed sets. That is, a monotonic function f maps a directed set S into the directed set $f(S)$, and we require $f(\sqcup S) = \sqcup f(S)$ for continuity.

2.2 Regular bases

For readers more familiar with continuous and algebraic lattices than with Scott domains we start with the following reminder.

The difference between a continuous Scott domain and a continuous lattice is that if $\top$ is a compact element, it can be omitted from the domain. Thus, for a continuous Scott domain A instead of the single top element $\top_A$ we have the set of maximal or total elements, $Total(A)$.

It is a part of standard theory, that $\forall x \in A.\ \exists y \in Total(A).\ x \sqsubseteq y$.

Now we introduce regular bases, which are useful in some situations. We say that the basis K is *regular* if $\forall k \in K, x \in Total(A).\ k \sqsubseteq x \Rightarrow k \ll x$.

Observe that a basis of compact elements of an algebraic Scott domain is always regular. Observe also that given an arbitrary continuous lattice A with basis K, it is very easy to obtain a regular basis K' by excluding those $k \in K$ for which $k \not\ll \top$. Unfortunately, some important domains, like the continuous domain of interval numbers, R^I, cannot be equipped with regular bases.

2.3 Continuous lattices of non-negative reals

We consider three basic domains of non-negative real numbers. Two of them include all non-negative real numbers and $+\infty$. The domain of lower estimates, R^+, has $\sqsubseteq=\leq$, in particular, $\perp = 0$ and $\top = +\infty$. The domain of upper estimates, R^-, has $\sqsubseteq=\geq$, in particular, $\perp = +\infty$ and $\top = 0$.

We call these, and all other totally ordered domains *vertical domains*. The informal meaning of $x \in R^+$ is that it is known that the number in question has the value of at least x, that is, we interpret $x \in R^+$ as the predicate "greater than or equal to x". The informal meaning of $x \in R^-$ is that it is known that the number in question has the value less than or equal to x. For this reason, we call them the domains of, respectively, lower and upper estimates.

Another important domain is the domain of interval non-negative reals, R^I. It consists of all intervals $[a, b]$, such that a and b are non-negative real numbers or $+\infty$, and $a \leq b$. The informal meaning of an interval number $x = [a, b]$ is that we know that the number in question is greater or equal to a and less or equal to b. $[a, b] \sqsubseteq_{R^I} [c, d]$ iff $a \leq c$ and $d \leq b$, which is the same as $[c, d] \subseteq [a, b]$. $\perp = [0, +\infty]$ and there is no top element in R^I. It might be useful to know that R^I can be obtained from $R^+ \times R^-$ by elimination of such pairs $< x, y >$, where $x > y$. That is, an interval estimate can be thought of as a combination of a lower estimate and an upper estimate, as long as the lower and upper estimates do not contradict each other.

All distances in this paper will have values in one of these three domains. Moreover, all distances will be between 0 and 1, although their sums may, of course, be larger than 1. In case of interval distances, all distances will be within the interval $[0, 1]$. All distances will obey some form of the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$. In the case of the domain of upper estimates one may wish to think about this property as the anti-triangle inequality, $d(x, y) + d(y, z) \sqsubseteq_{R^-} d(x, z)$. In the case of interval distances only upper bounds will obey the triangle inequality. The reasons for that are discussed in Subsection 5.5.

2.4 Measuring sets of basic elements

Consider the bottom element $\perp$ and an enumeration of all other elements of K, $k_1, ..., k_i, ...$. Associate weights with all basic elements: $w(\perp) = 0$, and let $w(k_i)$ form a converging sequence of strictly positive weights. For convenience we agree that the sum of weights of all basic elements equals 1. For example, one might wish to consider $w(k_i) = 2^{-i}$ or $w(k_i) = \epsilon(1 + \epsilon)^{-i}, \epsilon > 0$. Then the sum of weights of elements from any subset of K is well defined and belongs to $[0, 1]$. For $K_0 \subseteq K$ denote this sum as $W(K_0)$ and call it the weight of the subset K_0.

3 An asymmetric non-continuous distance

3.1 Definitions

Here we define a distance in the spirit of [3]. Define the distance between x and y in A as $d(x, y) = W(K_x \setminus K_y)$.

Define an open ball with the center $x \in A$ and real radius ϵ as $B_{x,\epsilon} = \{y \in A \mid d(x, y) < \epsilon\}$.

Define a subset B of A to be metric open if for any point $x \in B$ there is an $\epsilon > 0$ such that $B_{x,\epsilon} \subseteq B$.

3.2 Properties

Lemma 3.1 *$d(x, y) = 0$ iff $x \sqsubseteq y$.*

Lemma 3.2 (Triangle inequality) *$d(x, z) \leq d(x, y) + d(y, z)$.*

Theorem 3.1 *$B \subseteq A$ is metric open iff it is Scott open. Thus, d yields the Scott topology.*

3.3 Discussion

Unfortunately, there are many problems with $d(x, y)$ with regard to the intuition behind it and to the ability to compute it.

First, observe that $d(x, y) = 0$ iff $x \sqsubseteq y$. If one interprets $d(x, y)$ as some kind of cost of moving from x to y, one immediately would like to ask whether there is any deep reason behind the fact, that going up is free and going down is expensive, instead of vice versa. Of course, after a more careful look, one notices, that this depends on the arbitrary agreement to define the ball $B_{x,\epsilon}$ as the set $\{y \in A \mid d(x, y) < \epsilon\}$. If we instead define $B_{x,\epsilon} = \{y \in A \mid d(y, x) < \epsilon\}$, then we will have $d(x, y) = 0$ iff $y \sqsubseteq x$. Yet questions about the intuition behind the definitions of $d(x, y)$ and $B_{x,\epsilon}$ remain (see, however, Subsection 4.3).

Another problem is that whether one considers d to be $A \times A \to R^+$ or $A \times A \to R^-$, it is monotonic with respect to one argument and antimonotonic with respect to another. Thus, d is not Scott continuous, and it looks likely that there is no easy way to fix this. This is unpleasantly different from distance functions of classical metric spaces, and has grave negative implications for our ability to compute d, since the paradigm of domain theory requires computable functions to be continuous.

Let us look at the computability question more closely. First, consider an ideal and unrealistic situation — that the sets K_x and K_y are recursive. In fact, what we really need is a decision procedure for the question whether $k \in K_x \setminus K_y$ for $k \in K$. Then we can start with $[0, 1]$ as an approximation of $d(x, y)$ and for every $k \in K$ (assume that they are recursively enumerated) if $k \in K_x \setminus K_y$, then add $w(k)$ to the lower boundary, otherwise subtract $w(k)$ from the upper boundary. This way we can get as close to $[d(x, y), d(x, y)]$ as we want.

Unfortunately, the realistic situation in a typical application is that for a "reasonable" x, K_x is usually recursively enumerable, but it is rarely recursive. Let us illustrate this by considering the typical situation in semantics of programming languages. Consider a syntactic domain of programs, P, a semantic domain of meanings, S, and a continuous semantic function, $[\![\ ]\!] : P \to S$. Typically all domains are algebraic. Denote respective bases as K_P and K_S, and use K_x to denote the set of basic elements way below x, for both $x \in P$ and $x \in S$. Usually the syntactic domain is very simple, so if we consider a program $p \in P$, the set K_p is usually recursive. Now consider $[\![p]\!]$ and $K_{[\![p]\!]}$. Consider relation $\widehat{[\![\ ]\!]} = \{< k, k' > \in K_P \times K_S \mid k' \ll [\![k]\!]\}$. Sometimes $\widehat{[\![\ ]\!]}$ is recursive, although it is often only recursively enumerable. Now observe that $K_{[\![p]\!]} = \{k' \mid \exists k \in K_p.\ < k, k' > \in \widehat{[\![\ ]\!]}\}$. It is easy to see, that even under the weak assumptions that K_p and $\widehat{[\![\ ]\!]}$ are recursively enumerable, $K_{[\![p]\!]}$ will be recursive enumerable as well. However, even if both K_p and $\widehat{[\![\ ]\!]}$ are recursive, there are no reasons for $K_{[\![p]\!]}$ to be recursive.

Let us consider an even more specific example. In this paragraph we deal with Turing-complete languages. Consider the semantics expressing termination of programs with one numerical parameter. S will be the set of continuous functions $[N_\perp \to Void_\perp]$, where $N_\perp$ consists of incomparable natural numbers and a bottom element, and $Void_\perp$ is a two-element domain consisting of $\top$ and $\perp$. For any program p we will have $[\![p]\!](\perp) = \perp$. Also $[\![p]\!](n) = \top$ if p terminates when the input is n, $[\![p]\!](n) = \perp$ otherwise. Now recall a classical construction from the halting problem discourse. Consider a program I, which takes a number n and simulates application of a program coded by n to input n. $[\![I]\!](n) = \top$ iff the program coded by n terminates on n. It is well known that the halting problem is only semi-decidable, i.e. $\{n \mid [\![I]\!](n) = \top\}$ is recursively enumerable but not recursive. From this one can easily obtain that $K_{[\![I]\!]}$ is recursively enumerable but not recursive.

Thus, the only thing we can do efficiently is to subtract $w(k)$ from the upper boundary, whenever we discover that $k \in K_y$. Thus, we can at best get $[0, 1 - W(K_y)] = [0, W(K \setminus K_y)]$, which is a far cry from $[W(K_x \setminus K_y), W(K_x \setminus K_y)]$.

Of course, even if the problem whether $k \in K_x \setminus K_y$ is undecidable, we can have an *ad hoc* heuristic procedure which would decide this problem for many of the k's. This can allow us to achieve much better degree of approximation. On the other hand, each k, for which we fail to make the decision, will contribute $w(k)$ to the remaining gap (error).

There are three continuous functions somewhat related to $d(x, y)$. $f_1 : A \times A \to R^+$, $f_1(x, y) = W(K_x)$, $f_2 : A \times A \to R^-$, $f_2(x, y) = 1 - W(K_x \cap K_y)$, and $f_3 : A \times A \to R^-$, $f_3(x, y) = 1 - W(K_y)$. f_1 is, of course, *not* a valid lower boundary for $d(x, y)$, since the subsequent discovery that $k \in K_y$ would kill the contribution of k. f_2 and f_3 are both *valid* upper boundaries for $d(x, y)$. f_3 is more precise upper boundary than f_2, but it is still not ideal, since it does not account for those k, which do not belong to $K_x \cup K_y$. Of course, $d(x, y) = f_1(x, y) + f_2(x, y) - 1$.

We were hoping that by playing with combinations of f_1, f_2, and f_3 we would be able to make this situation somewhat better. To our great surprise, we discovered that f_2 alone can describe the Scott topology in a nice way!

4 A symmetric continuous generalized distance

4.1 Definitions

Define the distance between x and y in a continuous lattice A with basis K as $\rho(x, y) = 1 - W(K_x \cap K_y)$.

We will think about the function ρ as having the domain of upper estimates as its range: $\rho : A \times A \to R^-$.

An open ball is defined as in the previous section, only we use ρ instead of d. We keep the notation $B_{x,\epsilon}$ for an open ball.

We should formulate the notion of a metric open set a little more carefully than in the previous section, because it is now possible to have an empty ball of a non-zero positive radius (due to the fact that $\rho(x, x)$ is not always 0), and such balls are of no use.

Define a subset B of A to be metric open if for any point $x \in B$ there is an $\epsilon > 0$ such that $B_{x,\epsilon}$ is not empty and $B_{x,\epsilon} \subseteq B$.

4.2 Properties

Lemma 4.1 (Symmetry) *$\rho(x, y) = \rho(y, x)$.*

Lemma 4.2 *$\rho(x, y) = 0 \Rightarrow x = y = \top$. If K is a regular basis, $x = y = \top \Rightarrow \rho(x, y) = 0$.*

Remark: Recall that for a continuous lattice the basis can always be made regular.

Lemma 4.3 *$\rho(x, y) = 1$ iff $x \sqcap y = \bot$. In particular, $\rho(x, x) = 1$ iff $x = \bot$.*

Lemma 4.4 *If $x \sqsubseteq y$ then $\rho(x, y) = \rho(x, x)$. Otherwise, $\rho(x, y) > \rho(x, x)$.*

Lemma 4.5 (Strict monotonicity of the "norm") *If $x \sqsubset y$ then $\rho(x, x) > \rho(y, y)$ (i.e. $\rho(x, x) \sqsubset_{R^-} \rho(y, y)$).*

Lemma 4.6 (Monotonicity of the distance) *If $x \sqsubseteq y$ then $\rho(x, z) \geq \rho(y, z)$ (i.e. $\rho(x, z) \sqsubseteq_{R^-} \rho(y, z)$).*

Theorem 4.1 (Continuity of the distance) *Consider a directed set $B \subseteq A$ and some $z \in A$. Then $\rho(z, \sqcup B) = \sqcup_{R^-}\{\rho(z,x) \mid x \in B\} = \inf\{\rho(z,x) \mid x \in B\}$.*

Remark: This property is known as *Scott continuity* of the distance. As always in domain theory, continuity with respect to separate variables implies continuity with respect to all variables together.

Lemma 4.7 ("Anti"-triangle inequality) $\rho(x,y) + \rho(y,z) \sqsubseteq_{R^-} \rho(x,z)$ *(which is equivalent to $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$). In fact, Vickers-Matthews strong triangle inequality, $\rho(x,z) \leq \rho(x,y) + \rho(y,z) - \rho(y,y)$, also holds.*

Lemma 4.8 *The open ball $B_{x,\epsilon}$ is not empty iff $\epsilon > \rho(x,x)$. If this is the case, $B_{x,\epsilon}$ contains x and all y's, such that $x \sqsubseteq y$.*

Theorem 4.2 *$B \subseteq A$ is metric open iff it is Scott open. Thus, ρ yields the Scott topology.*

4.3 Discussion

The symmetric distance function $\rho(x,y)$ allows us to describe the asymmetric Scott topology by using monotonically decreasing $\rho(x,x)$ non-zero values. In particular, $\rho(\perp,\perp) = 1$ — the greatest possible distance in our "normalized" world. What is the intuition behind this distance function?

The best explanation seems to be as follows. Domain theory often thinks about an element x as a representative of all its possible refinements y ("y is a refinement of x" means that $x \sqsubseteq y$). $\rho(x,y)$ measures how far elements x and y can go from each other, if we independently *increase* them. Of course, $\rho(x,y)$ is an upper estimate of this — one should think about the meaning of $\rho(x,y)$ as $[0,\rho(x,y)]$. Moreover, if we consider approximations $x' \sqsubseteq x, y' \sqsubseteq y$, we obtain $\rho(x',y') \geq \rho(x,y)$ meaning that $[0,\rho(x',y')]$ approximates $[0,\rho(x,y)]$ in R^-. The dynamic interpretation of this is that when x and y become larger, $[0,\rho(x,y)]$ shrinks (it does not converge to 0, unless both x and y converge to $\top$), but $\rho(x,y)$ remains a correct upper estimate at every moment.

When $x = y$, one should still think about x and y as two *different* copies of the same partial element. We can increase these copies independently from each other, so that the equality is no longer preserved and these copies may become very distant from each other. One should think that $\rho(x,y)$ measures this potential divergence even when $x = y$. The following analogy with interval numbers might help to clarify this: $[-2,2]^2 = [0,4]$, but $[-2,2] \times [-2,2] = [-4,4]$. Of course, in the second case, the left $[-2,2]$ may approximate some point x, and the right $[-2,2]$ may approximate some point y, and x does not have to be equal to y. The fact that $[-2,2] \times [-2,2]$ contains negative numbers breaks conventional intuition in exactly the same way and via essentially the same mechanism, as the fact that $\rho(x,x) > 0$ does. Also see the Introduction for an example of this type of distance on interval numbers.

This distance was found during the search for a metric-like structure which would satisfy the triangle inequality and Theorems 4.1, 4.2. Recently Bob Flagg helped us to realize that ρ is a Matthews partial distance [5]. We would like to observe that the equality $d(x,y) = \rho(x,y) - \rho(x,x)$ holds and serves as a manifestation of Theorem 4.1 of [5]. Thus Lemma 4.4 implies Lemma 3.1, finally explaining some of the intuition behind $d(x,y)$.

It turns out that our construction closely resembles one by O'Neill [6]. It is also valuation based, but we consider continuous valuations on the powerset of K instead of continuous valuations on the domain itself, so our spaces are not O'Neill valuation spaces. The use of the valuations on the domain itself limits the approach in [6] to the domains having prime bases P due to its reliance on the $P_x \cup P_y = P_{x \sqcup y}$ equality. We also avoid negative-valued distances of [6]. The dual partial metric in the style of [6] would be expressed by a slightly different formula:

$\rho^*(x,y) = 1 + \rho(x,y) - \rho(x,x) - \rho(y,y) = W(K_x \cup K_y)$. The induced metric would be expressed as $d(x,y) + d(y,x) = \rho(x,y) + \rho^*(x,y) - 1 = 2 * \rho(x,y) - \rho(x,x) - \rho(y,y)$, similar to the *weighted metric* of [5].

The actual computation of $\rho(x,y)$ goes as follows. Start with $[0,1]$ and go along the recursive enumerations of K_x and K_y. Whenever we discover, that some k occurs in both K_x and K_y, subtract $w(k)$ from the upper boundary. If this process continues long enough, $[0, \rho(x,y)]$ is approximated as well as desired.

Unfortunately, in this case there is no general way to compute meaningful lower estimate for $\rho(x,y)$. Consequently, there is no general way to determine how close is the convergence process to the actual value of $\rho(x,y)$.

5 Generalization to the continuous Scott domains

5.1 Lower estimates

Consider the definition of ρ applied to the continuous Scott domains. We claim, that all results of the previous section still hold.

Thus, if the domain does not have a top element, $\rho(x,y)$ never equals to 0. We would like to be able to compute lower estimates of ρ in the spirit of the discussion at the end of the previous section. More specifically, we would like to introduce the function $\tau(x,y)$ such that the following condition holds.

Condition 5.1 $\forall x', y' \in A.(x \sqsubseteq x' \& y \sqsubseteq y') \Rightarrow \tau(x,y) \leq \rho(x', y')$.

It is clear that if $x \sqcup k$ or $y \sqcup k$ does not exist for some basic element k, then k will never make it into $K_{x'} \cap K_{y'}$, and $w(k)$ will contribute to $\rho(x', y')$ for all x', y' in question. Denote $I_x = \{k \in K \mid k \sqcup x \text{ does not exist}\}$. We define $\tau(x,y) = W(I_x \cup I_y)$.

We claim that $\tau(x,y)$ is a monotonic, Scott continuous function into R^+, that Condition 5.1 holds, and that the following condition is true.

Condition 5.2 $\forall x, y \in A. \tau(x,y) = \rho(x,y) \Rightarrow x, y \in Total(A)$. *If K is a regular basis, the inverse implication is true as well.*

Actually we can view $< x, y > \mapsto [\tau(x,y), \rho(x,y)]$ as a Scott continuous function to interval numbers.

Since the "anti"-triangle rule holds for ρ, and in view of Condition 5.2, the restriction of $< \tau, \rho >$ onto $Total(A)$ can almost be viewed as a classical metric if K is regular. The only obstacle is the fact, that $\rho(x,x) \neq 0$ even for $x \in Total(A)$. Below we present a modified construction of $< \tau, \rho >$ which fixes this obstacle.

5.2 Definitions

In the section on basic notions we introduced interval real numbers, R^I, and explained the relationship between R^I and $R^+ \times R^-$.

We define the new notion of distance, $\delta : A \times A \to R^I$, where $\delta(x,y) = [l(x,y), u(x,y)]$, as follows.

$u(x,y) = \rho(x,y) - W(I_x \cap I_y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y)$.

$l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x)$.

An open ball with the center $x \in A$ and real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x,y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes points y, about which we are *sure* that they are not too far from x.

Metric open sets are defined via open balls as in the previous section.

5.3 Properties

Lemma 5.1 (δ is a conservative generalization of ρ) *If A has the top element, $\top$, then $\delta(x,y) = [0, \rho(x,y)]$.*

Lemma 5.2 (Symmetry) *$\delta(x,y) = \delta(y,x)$.*

Lemma 5.3 *$K_x \cap I_x = \emptyset$.*

Lemma 5.4 *$K_x \cup I_x = K \Rightarrow x \in Total(A)$. If K is a regular basis, the inverse implication is also true.*

Proof: If every $k \notin K_x$ belongs to I_x, then it is obvious that x is maximal.

Now assume that $x \in Total(A)$. Then if $k \notin I_x$, then $k \sqsubseteq x$. Now use the regularity of K and obtain $k \ll x$ and $k \in K_x$. □

Lemma 5.5 (Correctness of lower estimate) *$l(x,y) \leq u(x,y)$.*

Proof: Using Lemma 5.3 we can rewrite u and l. $u(x,y) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) = W(\overline{U})$, where $U = (K_x \cap K_y) \cup (I_x \cap I_y)$. $l(x,y) = W(V)$, where $V = (K_x \cap I_y) \cup (K_y \cap I_x)$.

We want to show that $V \subseteq \overline{U}$. For that it is enough to show that $V \cap U = \emptyset$. We show that $(K_x \cap I_y) \cap U = \emptyset$. Then by symmetry the same will be true for $K_y \cap I_x$, and hence for V.

$(K_x \cap I_y) \cap U = (K_x \cap I_y \cap K_x \cap K_y) \cup (K_x \cap I_y \cap I_x \cap I_y)$. But $K_x \cap I_y \cap K_x \cap K_y \subseteq I_y \cap K_y = \emptyset$. Similarly, $K_x \cap I_y \cap I_x \cap I_y = \emptyset$. □

Lemma 5.6 *$l(x,y) = u(x,y)$ implies that both x and y are maximal elements. If K is a regular basis, the inverse implication is also true.*

Proof: Assume first that x and y are maximal elements and K is regular. Using the notations of the previous proof we want to show that $\overline{U} \subseteq V$. Using Lemma 5.4, we have $K_x \cup I_x = K_y \cup I_y = K$.

If $k \notin U$, then $k \notin K_x$ or $k \notin K_y$. Because of the symmetry it is enough to consider $k \notin K_x$. Then $k \in I_x$. Then, using $k \notin U$ once again, $k \notin I_y$. Then $k \in K_y$. So we have $k \in K_y \cap I_x \subseteq V$.

Assume now that $\overline{U} \subseteq V$ and x is not a maximal element. Let us find $k \notin U$ such that $k \notin V$. Consider a maximal element x' such that $x \sqsubset x'$, and take $k \in K_{x'}$, $k \notin K_x$. $k \in K_{x'}$ implies $k \notin I_x$, which together with $k \notin K_x$ implies $k \notin U$, $k \notin V$ — contradiction. □

Lemma 5.7 *$\delta(x,x) = [0,1]$ iff $u(x,x) = 1$ iff $x = \bot$.*

Lemma 5.8 (Monotonicity) *If $x \sqsubseteq y$ then $K_x \subseteq K_y$ and $I_x \subseteq I_y$, which implies $u(x,z) \geq u(y,z)$ and $l(x,z) \leq l(y,z)$, i.e. $\delta(x,z) \sqsubseteq_{R^I} \delta(y,z)$.*

Lemma 5.9 (Strict monotonicity of upper estimate of the "norm")
If $x \sqsubset y$ then $u(x,x) > u(y,y)$ and $\delta(x,x) \sqsubset_{R^I} \delta(y,y)$.

Lemma 5.10 *If $x \sqsubseteq y$ then $l(x,y) = 0$ and $u(x,y) = u(x,x)$, which implies $\delta(x,y) = \delta(x,x)$. If $x \not\sqsubseteq y$ then $u(x,y) > u(x,x)$.*

Lemma 5.11 (Triangle inequality for upper estimate) *$u(x,y) + u(y,z) \geq u(x,z)$. Vickers-Matthews strong triangle inequality, $u(x,y) + u(y,z) - u(y,y) \geq u(x,z)$, also holds.*

Proof: We want to show $1 - W(K_x \cap K_z) - W(I_x \cap I_z) \leq 1 - W(K_x \cap K_y) - W(I_x \cap I_y) + 1 - W(K_y \cap K_z) - W(I_y \cap I_z)$. This is equivalent to $W(K_x \cap K_y) + W(K_y \cap K_z) + W(I_x \cap I_y) + W(I_y \cap I_z) \leq 1 + W(K_x \cap K_z) + W(I_x \cap I_z)$.

Notice, that $W(K_x \cap K_y) + W(K_y \cap K_z) = W(K_x \cap K_y \cap K_z) + W(K_y \cap (K_x \cup K_z))$, and the similar formula holds for I's.

Now $W(K_x \cap K_y \cap K_z) \leq W(K_x \cap K_z)$, and the similar inequality holds for I's.

Hence it is enough to show, that $W(K_y \cap (K_x \cup K_z)) + W(I_y \cap (I_x \cup I_z)) \leq 1$. Since $K_y \cap I_y = \emptyset$, we have $W(K_y \cap (K_x \cup K_z)) + W(I_y \cap (I_x \cup I_z)) = W(K')$, where $K' = (K_y \cap (K_x \cup K_z)) \cup (I_y \cap (I_x \cup I_z))$. But for any set K' of basic elements, $W(K') \leq 1$.

This proves the triangle inequality, and it is easy to modify the proof to obtain the Vickers-Matthews version. □

Theorem 5.1 (Continuity of the distance) *Consider a directed set $B \subseteq A$ and some $z \in A$. Then $u(z, \sqcup B) = \inf\{u(z,x) \mid x \in B\}$ and $l(z, \sqcup B) = \sup\{l(z,x) \mid x \in B\}$, which implies that $\delta(z, \sqcup B) = \sqcup_{R^I}$.*

Proof: We want to show that $W(K_z \cap K_{\sqcup B}) + W(I_z \cap I_{\sqcup B}) = \sup\{W(K_z \cap K_x) + W(I_z \cap I_x) \mid x \in B\}$ and $W(K_z \cap I_{\sqcup B}) + W(I_z \cap K_{\sqcup B}) = \sup\{W(K_z \cap I_x) + W(I_z \cap K_x) | \; x \in B\}$. Monotonicity considerations trivially yield both "$\geq$" inequalities, so it is enough to show "$\leq$" inequalities. In fact, we will show that for any sets $C \subseteq A$ and $D \subseteq A$, $W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B}) \leq \sup\{W(C \cap K_x) + W(D \cap I_x) \mid x \in B\}$ holds.

It is easy to show that $K_{\sqcup B} = \cup\{K_x \mid x \in B\}$ by showing first that the set $\cup\{K_x \mid x \in B\}$ is directed and $\sqcup B = \sqcup(\cup\{K_x \mid x \in B\})$. Let us prove that $I_{\sqcup B} = \cup\{I_x \mid x \in B\}$. "$\supseteq$" is trivial. Let us prove "$\subseteq$". Assume that $k \notin \cup\{I_x \mid x \in B\}$, i.e. $\forall x \in B \; k \sqcup x$ exists. It is easy to see that because B is a directed set, $\{k \sqcup x \mid x \in B\}$ is also directed. Then $k \sqsubseteq \sqcup\{k \sqcup x \mid x \in B\} \sqsupseteq \sqcup B$, implying existence of $k \sqcup (\sqcup B)$ and, hence, $k \notin I_{\sqcup B}$.

Now consider enumerations of the countable or finite sets $K_{\sqcup B}$ and $I_{\sqcup B}$: $k_1, ..., k_n, ...$ and $k'_1, ..., k'_n, ...$, respectively. Define tail sums $S_n = w(k_n) + w(k_{n+1}) + ...$ and $S'_n = w(k'_n) + w(k'_{n+1}) + ...$ Observe that (S_n) and (S'_n) converge to 0.

Pick for every k_n some $x_n \in B$ such that $k_n \in K_{x_n}$. Pick for every k'_n some $x'_n \in B$ such that $k'_n \in I_{x'_n}$. Then using the directness of B, we can for any n pick such $y_n \in B$, that $x_1, ..., x_n, x'_1, ..., x'_n \sqsubseteq y_n$. Then $k_1, ..., k_n \in K_{y_n}$ and $k'_1, ..., k'_n \in I_{y_n}$. It is easy to see that $(W(C \cap K_{\sqcup B}) + W(D \cap I_{\sqcup B})) - (W(C \cap K_{y_n}) + W(D \cap I_{y_n})) = W(C \cap K_{\sqcup B}) - W(C \cap K_{y_n}) + W(D \cap I_{\sqcup B}) - W(D \cap I_{y_n}) < S_n + S'_n$, which can be made as small as we like. □

Lemma 5.12 *The open ball $B_{x,\epsilon}$ is not empty iff $\epsilon > u(x,x)$. If this is the case, $B_{x,\epsilon}$ contains x and all y's, such that $x \sqsubseteq y$.*

Theorem 5.2 *$B \subseteq A$ is metric open iff it is Scott open. Thus, δ yields the Scott topology.*

Proof: Assume that B is metric open. Take $x \in B$. There is a ball $B_{x,\epsilon} \subseteq B$, $x \in B_{x,\epsilon}$, and this ball is the inverse image of the open set $[0, \epsilon) \subseteq R^-$ under the Scott continuous function $f(y) = u(x,y) : A \to R^-$. Hence, this ball is Scott open, and when we apply this construction to all $x \in B$, we obtain that B is a union of such balls which are Scott open sets. Hence B is Scott open.

Conversely, assume that B is Scott open. Take $x \in B$. Because B is Scott open there is a basic element $k \in B$ such that $k \ll x$. We must find $\epsilon > 0$ such that $x \in B_{x,\epsilon} \subseteq B$. Let $\epsilon = u(x,x) + w(k)/2$. Clearly $x \in B_{x,\epsilon}$. We claim that $B_{x,\epsilon} \subseteq \{y | y \gg k\} \subseteq B$. Assume, by contradiction, that $y \gg k$ is false. Then $k \notin K_y$ and $k \in K_x$. Then $W(K_x \cap K_y) + W(I_x \cap I_y) + w(k) \leq W(K_x) + W(I_x)$. Then $u(x,y) - w(k) = 1 - W(K_x \cap K_y) - W(I_x \cap I_y) - w(k) \geq 1 - W(K_x) - W(I_x) = u(x,x)$. Therefore $u(x,y) \geq u(x,x) + w(k)$ and thus $y \notin B_{x,\epsilon}$. □

5.4 Subspace topology and $Total(A)$

Consider a domain A with a generalized distance δ. Consider $B \subseteq A$ and the restriction of δ onto $B \times B$. Let us retain the definition of the metric open set for this new $\delta : B \times B \to R^I$ using, of course, only points from B. We denote balls taken within B as $B'_{x,\epsilon}$ and retain the notation $B_{x,\epsilon}$ for the balls taken within A. It is easy to show that these metric open sets form a topology, and that this topology is precisely the subspace topology of the metric topology on A.

Now assume that our space has a regular basis and, thus, Lemmas 5.4 and 5.6 hold. Consider the restriction of δ onto $B = Total(A)$. For every $x, y \in Total(A)$ $l(x,y) = u(x,y)$ and, hence, $\delta(x,y) = [l(x,y), l(x,y)] = [l(x,y), u(x,y)] = [u(x,y), u(x,y)]$. That is, distances on total elements are total interval numbers and can be considered as ordinary numbers: $\delta' : Total(A) \times Total(A) \to R$, $\delta'(x,y) = l(x,y) = u(x,y)$. Then $Total(A)$ becomes an ordinary metric space with metric δ' (consider $\delta'(x,y) = u(x,y)$ to prove that), and this metric space defines the subspace topology of the Scott topology on A. Now use $\delta'(x,y) = l(x,y)$ and observe that by Lemma 5.4 $I_x = K \setminus K_x$ if x is total. Then $l(x,y) = W(K_x \cap I_y) + W(K_y \cap I_x) = W(K_x \setminus K_y) + W(K_y \setminus K_x) = d(x,y) + d(y,x)$, and we obtain $\delta'(x,y) = d(x,y) + d(y,x)$, where d is our version of the Kopperman-Flagg distance.

5.5 Discussion

The actual computation of $\delta(x,y)$ goes as follows. Start with $[0,1]$ and go along the recursive enumerations of K_x, K_y, I_x, and I_y. Whenever we discover that some k occurs in both K_x and K_y, or in both I_x and I_y, subtract $w(k)$ from the upper boundary. Whenever we discover that some k occurs in both K_x and I_y, or in both I_x and K_y, add $w(k)$ to the lower boundary. If this process continues long enough, $[l(x,y), u(x,y)]$ is approximated as well as desired.

However, there is no general way to compute a better lower estimate for $u(x,y)$ than $l(x,y)$, or to compute a better upper estimate for $l(x,y)$ than $u(x,y)$. Consequently, there is no general way to determine how close is the convergence process to the actual values of $l(x,y)$ and $u(x,y)$, except that we know that $u(x,y)$ is not less than currently computed lower bound, and $l(x,y)$ is not greater than currently computed upper bound. Of course, for large x and y this knowledge can provide a lot of information, and if the basis of our domain is regular, for total x and y this knowledge provides us with precise estimates — i.e. if the basis is regular, then the resulting classical metric on $Total(A)$ can be nicely computed.

Our definition of open balls shows that upper bounds play much more important role than lower bounds. One can also observe that we proved the triangle inequality for upper bounds, but not for lower bounds. In fact, there are deep reasons for the triangle inequality to hold for upper bounds, but not for lower bounds. Assume that $u(x,y)$ is in fact an upper bound for some hypothetical "genuine distance", $\sigma(x,y)$, for which the triangle inequality, $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$, holds. Then if $u(x,y) > u(x,z) + u(z,y)$, then $u(x,z) + u(z,y)$ gives a better upper estimate for $\sigma(x,y)$ than $u(x,y)$. This means that unless $u(x,y) \leq u(x,z) + u(z,y)$, u could be easily improved and, hence, would be very imperfect.

This kind of reasoning is not valid for $l(x,y)$. Indeed, assume that $l(x,y)$ is a lower bound for some ideal distance $\sigma(x,y)$. But the triangle inequality for σ does not yield *separate* lower estimates for $\sigma(x,z)$ and $\sigma(z,y)$, i.e. it does not say anything about $l(x,z)$ and $l(z,y)$. In fact, it is quite possible that $l(x,y) \neq 0$, but $l(x,z) = l(z,y) = 0$. For example, recall our distance on interval numbers described in the Introduction. The distance between $[2,2]$ and $[3,3]$ is $[1,1]$, but the distances between $[2,2]$ and $[2,3]$ and between $[2,3]$ and $[3,3]$ are equal to $[0,1]$. Thus the lower estimate of the distance between $[2,2]$ and $[3,3]$ is 1, but the lower estimates of the distances between $[2,2]$ and $[2,3]$ and between $[2,3]$ and $[3,3]$ are equal to 0. The reason for this is that $[2,2]$ and $[3,3]$ are *different refinements* of $[2,3]$ themselves.

We see two problems with our current construction. First, it does not incorporate some natural examples, like the example of the distance on interval numbers in the Introduction, mainly due to the excessive reliance on countable bases. Second, it is very sensitive to such operations as an addition of a new top element to a domain - the distance in [3] does not have this problem. This problem comes from focusing too much on $Total(A)$. We hope to be able to overcome these two problems via generalizing the ideas of this paper and [6].

In this context, we should note that [5] advocates the focus on $Total(A)$. This is unlikely to work well in the applications we have in mind, because many important correct programs are inherently partial. In fact any program using its arguments directly or due to call-by-value mechanism is likely not to denote a total element. For example, the identity function is usually not maximal in $[D \to D]$, unless D obeys very special restrictions.

6 Conclusion

Partial metrics yield a foundation needed in order to compute meaningful distances between programs. They represent a significant step towards understanding programs from the viewpoint of continuous mathematics.

The next natural step is to do case studies of partial metrics for some of the already known examples of denotational semantics for specific programming languages in order to find out, whether it is possible to obtain distances which "look right" for particular pairs of programs.

On the other hand, we do not see how having a metric structure on programs can by itself lead to truly significant applications. More classical functional analysis needs to be imported into the setting of domains in order to obtain such applications. We always considered measure theory together with the ability to integrate and to take pseudo-random elements according to some distribution from a reasonable neighborhood as the first candidate for such import. We hope that recently emerging applications of domains to measure theory (see [1]), and the widespread use of valuations indicate that applications of measure theory to domains will be soon within reach. If this goal is achieved, a lot of algorithms from numerical analysis including some optimization methods will become immediately applicable to sets of programs.

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