Partial Metrics and Quantale-valued Sets (Preprint)

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Abstract

It is observed that the axioms for partial metrics with values in quantales coincide with the axioms for Q-sets (M-valued sets, sets with fuzzy equality, quantale-valued sets) for commutative quantales. Ω-sets correspond to the case of partial ultrametrics.

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1 Introduction

During our studies of a Ph.D. thesis by Kim Wagner [10], we observed that Ω-sets strongly resemble partial ultrametrics. We followed the references given in [10] in search for possible generalizations from complete Heyting algebras to more general commutative quantales.

On page 275 of a textbook [11] by Oswald Wyler, we found a definition of the notion of fuzzy equality, which coincided with the definition of partial metrics with values in quantales in [5] modulo notation and some particulars of the restrictions imposed on the quantales in question.

This definition was originally given by Ulrich Höhle in [3], where a set equipped with fuzzy equality was called an M-valued set. The paper [3] generalized the treatment of sheaves via Ω-sets (sets with generalized equality valued in a complete Heyting algebra), introduced by Michael Fourman and Dana Scott in their seminal paper [2].

The purpose of the present short paper is to report this remarkable coincidence and to briefly describe the relevant context.

1.1 Domains and sheaves

Domains for denotational semantics were introduced by Dana Scott in the late 1960-s to solve reflexive domain equations and to give denotational semantics to programming languages such as lambda calculus [8]. Mathematical history of sheaves goes back to at least 1940-s and is beyond the scope of this paper.

Domains and sheaves represent different approaches to the theory of partially defined elements. While certain ideological affinity between these approaches is recognized, domains and sheaves are usually treated as technically unrelated.

We expect that the striking coincidence between the notion of generalized equality (fuzzy equality) in the context of sheaves and the notion of generalized symmetric distance (partial metric) in the context of domains will lead to tighter connections between sheave-based and domain-based approaches to the theory of partially defined elements.

2 Partial metrics

The generalized distances without the axiom $p(x, x) = 0$ in the context of analyzing deadlock in lazy data flow computations were studied by Steve Matthews in his Ph.D. thesis [6]. Then certain axioms regaining some of what is lost by dropping $p(x, x) = 0$ were added, namely, small self-distances, $p(x, x) \leq p(x, y)$, and the sharpened form of triangular equality, $p(x, z) \leq p(x, y)+p(y, z)-p(y, y)$, introduced by Steve Vickers in [9]. The canonical first publication describing partial metrics as we know them today is [7].

Michael Bukatin and Joshua Scott studied generalized distances on Scott domains and noted that axiom $p(x, x) = 0$ is incompatible with Scott continuity (or computability) of distances in question [1]. It should be noted that in the similar fashion the axiom $x = x$, which can be rewritten as $Eq(x, x) = true$, prevents the traditional equality from being Scott continuous (or computable).

Because Bukatin and Scott were interested in Scott continuity of the resulting distances and, hence, needed monotonicity, their domain of numbers was the set of non-negative reals (with added $+\infty$) with the reverse order: 0 was the largest element, and $+\infty$ was the smallest.

However, the traditional view of 0 as the smallest possible distance remained more prevalent in the partial metrics research community, and when Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh generalized partial metrics so that they would take their values in quantales rather than in non-negative reals [5], the axioms looked as follows.

The quantale V in question was a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element was the bottom element 0. The right adjoint to the map $b \mapsto a+b$ was defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a+c \geq b\}$. Certain additional conditions were imposed.

The axioms for a partial pseudometric (V-pseudopmetric) $p : X \times X \to V$ were

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

The separation axiom, making p into a partial metric (V-pmetric), was written as $\forall x, y \in X. x = y$ iff $p(x,y) = p(x,x) = p(y,y)$.

For a useful collection of publications related to partial metrics see URL: http://www.dcs.warwick.ac.uk/pmetric/pub.html.

3 Quantale-valued sets

3.1 Ω-sets

Ω-sets were introduced by Fourman and Scott in [2]. If Ω is a complete Heyting algebra, an Ω-set A is a set equipped with an Ω-valued generalized equality, $E : A \times A \to \Omega$, subject to axioms $E(a,b) = E(b,a)$ and $E(a,b) \wedge E(b,c) \leq E(a,c)$.

The paper [2] also introduced a mechanism of *singletons*, which was used to define the notion of *complete* Ω*-set* and to establish that complete Ω-sets and sheaves over complete Heyting algebra Ω are essentially the same thing.

3.2 Partial ultrametrics

Returning to partial metrics, from the symmetry and the ultrametric triangle inequality, $p(x,z) \leq max(p(x,y), p(y,z))$, one can obtain $p(x,x) \leq max(p(x,y), p(y,x)) = p(x,y)$.

$p(x,z) \leq max(p(x,y), p(y,z))$ means $p(x,z) \leq p(x,y)$ or $p(x,z) \leq p(y,z)$. Consider $p(x,z) \leq p(x,y)$. We know that $p(y,y) \leq p(y,z)$, so $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$, and we obtain the Vickers form of triangle inequality. Consider $p(x,z) \leq p(y,z)$. We know that $p(y,y) \leq p(x,y)$, and we again obtain $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

So utrametrics without axiom $p(x,x) = 0$ obey both extra axioms of partial metrics. This justifies the term *partial ultrametrics* and also tells us that we should consider Ω-equality as partial ultrametric with more general values than non-negative reals.

3.3 M-valued sets/fuzzy equality

This motivates the search for generalizations of Ω-equality beyond complete Heyting algebras. As we mentioned in the introduction, this search led us to [3] via [11] and [10].

In his work [3], Ulrich Höhle was motivated by the need to give solid foundation to fuzzy set theory (and, in particular, to the uses of such logical systems as Lukasiewicz logic). His definition of an M-valued set looked as follows.

The quantale V in question was a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element was the top element 1. The right adjoint to the map $b \mapsto a * b$ was defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \leq b\}$. Certain additional conditions were imposed.

An M-valued set was a set X equipped with a map $E : X \times X \to M$ subject to the axioms

- $E(x, y) \leq E(x, x) \wedge E(x, y)$
- $E(x, y) = E(y, x)$
- $E(x, y) + (E(y, y) \Rightarrow E(y, z)) \leq E(x, z)$

It's easy to see that the only difference between an M-valued set and a set with a V-pseudopmetric, besides the particular restrictions imposed on the quantale, is in notation: multiplicative vs. additive, the adjoint operation is denoted differenty and the order of its arguments is switched, and the quantale order is reversed.

So the notions of an M-valued set and a set with a V-pseudopmetric coincide.

This is the main observation we would like to communicate in the present paper.

3.4 Further observations

An M-valued set is called separated iff the axiom

$$E(x, x) \vee E(y, y) \leq E(x, y) \text{ implies } x = y$$

holds. It's easy to see that this is equivalent to the separation axiom for V-pmetric.

There are further parallels, e.g. similar extra conditions on quantales are imposed and studied, such as existence of *halves* in V vs. existence of *square roots* in M, etc.

We should also say that Höhle provides a generalization of singletons from [2] and develops a comprehensive theory relating M-sets to sheaves in [3].

For the current state of the quantale-valued sets, including generalizations to non-commutative quantales, see [4] and references therein.

4 Discussion

We are at the beginning of the process of looking at all parallels between the two approaches described in the present paper and we are assessing the possible implications of the coincidence we observed.

However, the theory of partial metrics (including partial metrics on domains) and the theory of quantale-values sets (with its specific relation to fuzzy sets and to sheaves) are both rich, well developed theories in their own right, and we should therefore expect that their new interaction will be quite fruitful.

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