Looking for common patterns

Michael Bukatin

Nokia Corporation, Cambridge, MA

Conference on generalized metrics for limits, computing, and more
Medgar Evers College of CUNY, April 10, 2013

Electronic coordinates

These slides are avalable from:

http://www.cs.brandeis.edu/∼bukatin/partial_inconsistency.html

E-mail:

bukatin@cs.brandeis.edu

Abstract

Progress often comes from identifying common patterns and motives in different mathematical and natural phenomena.

In this talk we give some examples of such common patterns.

In particular, we focus on the pattern of "strong triangularity", which occurs in various contexts including partial metrics and fuzzy equalities, and on the "bilattice pattern", which is known to occur in the contexts of non-monotonic reasoning, partial contradictions, bitopology, and probabilistic programming.

Outline

1. Introduction

2. Strong triangularity

3. Bilattice pattern

transitivity, triangularity, composition, ...

$d(x, y) + d(y, z) \leq d(x, z)$

$a \leq b \ \& \ b \leq c \Rightarrow a \leq c$

$a \to b \ \& \ b \to c \Rightarrow a \to c$

$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$

$\mathbb{A}(z, y) \circ \mathbb{A}(y, x) \leq \mathbb{A}(z, x)$

...

Natural sciences

Eugene Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences, Richard Courant lecture in mathematical sciences delivered at New York University, May 11, 1959, *Communications on Pure and Applied Mathematics* **13**, 1–14, 1960.

I think this unreasonable effectiveness to a considerable extent comes from the tendency of various aspects of nature to form and exhibit various classes of similar patterns.

Generalization and transfer; categories

Categories, enriched categories, abstract data types, ...

The effeciency of the theory of categories, in particular, comes from their superefficiency of describing and studying various common patterns.

Great for studies of high general applicability. Great for transfer of knowledge between various fields. Very productive at the high end.

Not so great: high abstraction barrier, high barriers to entry, loss of intuition and of expressive notation from particular fields, different patterns look alike, the differences between them get blurred.

Can we do something to counter the "not so great" part?

Non-standard generalization; individuation

We should try to generalize by importing notions into unusual frameworks (e.g. into the metric framework instead of the categorial framework) just to get different flavors of representations and to get viewpoints from different angles.

Individual treatment of particular patterns: use non-standard notation, distinctive fonts, less standard geometric presentations and icons (categorial diagrams are great, but one eventually gets tired of them being everywhere).

Strong triangularity ("Vickers form") for partial metrics

Steve Vickers discovered the following strengthening of the triangularity axiom for metrics with non-zero self-distances discovered by Steve Matthews:

$$p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$$

or equivalently,

$$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

In quantales

For quantale-valued fuzzy equalities (Höhle, 1992):

$$E(x, y) * (E(y, y) \Rightarrow E(y, z)) \sqsubseteq E(x, z)$$

For quantale-valued partial metrics
(Kopperman, Matthews, Pajoohesh, 2004):

$$p(x, z) \leq p(x, y) + (p(y, z) \dot{-} p(y, y))$$

Further appearances of this pattern

Strong trianularity axiom for partial metric:
$p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$.

Obtaining weighted metric from partial metric:
$d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$.

Metric-entropy pairs on lattices (Simovici, 2007):

Definition: (d, η) is a $\wedge$-pair if
$d(x,y) = 2\eta(x \wedge y) - \eta(x) - \eta(y)$.

Theorem: for a $\wedge$-pair $d(x,y) \leq d(x,z) + d(z,y)$ iff
$\eta(z) + \eta(x \wedge y) \leq \eta(x \wedge z) + \eta(y \wedge z)$.

An explanation for the pattern

There are many explanations for why
$p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$
looks this way.

The best explanation I know comes from the theory of categories enriched in quantaloids and states that the operaton $a \circ_y b = a + b - p(y,y)$ should be defined in such a way, that $p(y,y)$ is a unit of this operation, even if $p(y,y) \neq 0$.

Namely, we want $a \circ_y p(y,y) = a$ and $p(y,y) \circ_y b = b$.

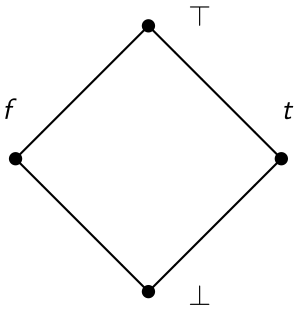
I think we'll see more appearances of this pattern in various situations.
Can we create software which would search for this pattern?

Bilattice pattern

This is can also be viewed as a short preview of tomorrow's talk.

I omit most references to the literature today and include them in the slide deck I'll be using tomorrow.

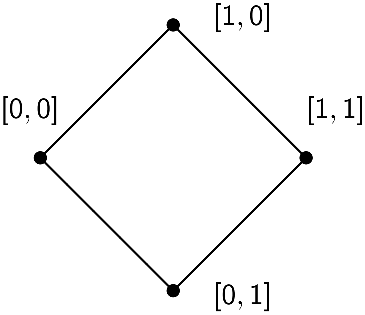
Example of a bilattice



$f < \bot < t, f < \top < t$

$\bot \sqsubset f \sqsubset \top, \bot \sqsubset t \sqsubset \top$

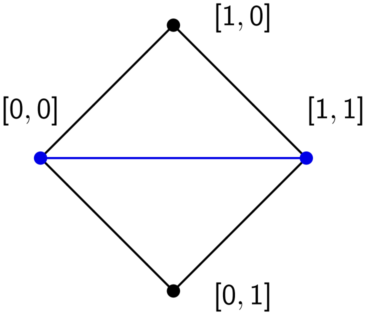
Partially inconsistent interval numbers within a segment



$[a, b] \leq [c, d]$ iff $a \leq c, b \leq d$

$[a, d] \sqsubseteq [b, c]$ iff $a \leq b, c \leq d$

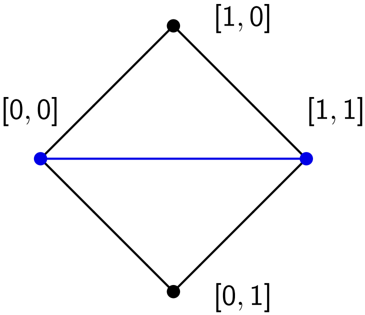
Partially inconsistent interval numbers within a segment



$[a, b] \leq [c, d]$ iff $a \leq c, b \leq d$

$[a, d] \sqsubseteq [b, c]$ iff $a \leq b, c \leq d$

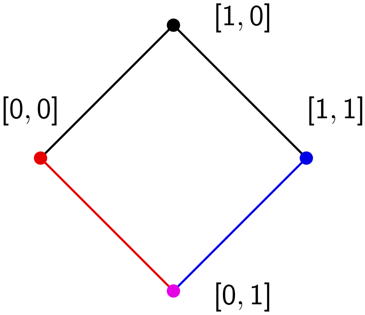
Partially inconsistent interval numbers within a segment



blue – precisely defined numbers

pseudosegments are above the blue

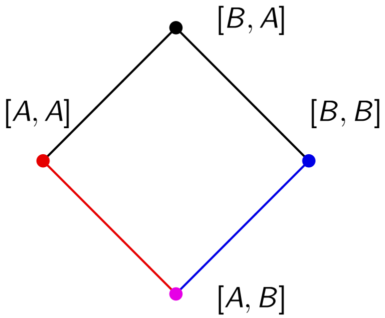
Negative and positive subspaces



Negative – space of upper bounds $[0, x]$

Positive – space of lower bounds $[x, 1]$

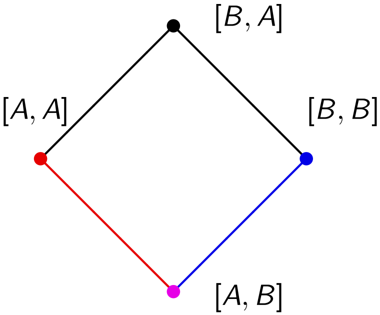
Negative and positive subspaces



Negative – space of upper bounds $[A, x]$

Positive – space of lower bounds $[x, B]$

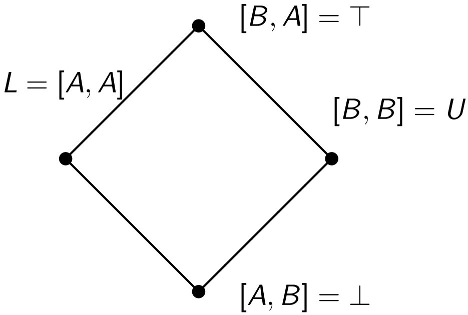
Negative and positive subspaces



We require $A < B$.

We can even allow $A = -\infty, B = \infty$.

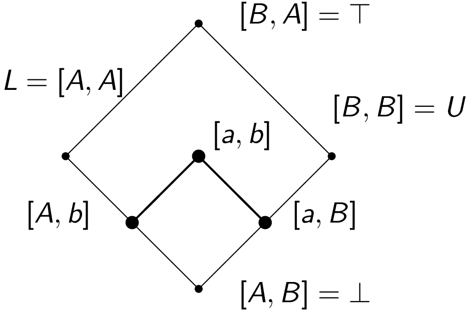
Negative and positive subspaces



We require $A < B$.

We can even allow $A = -\infty, B = \infty$.

Decomposition into negative and positive subspaces



$[A, b] = [a, b] \wedge \bot = [a, b] \sqcap L$
$[a, B] = [a, b] \vee \bot = [a, b] \sqcap U$
$[a, b] = [A, b] \sqcup [a, B]$

The idea of bilattice

Matthew L. Ginsberg. Multivalued logics: a uniform approach to inference in artificial intelligence. *Computational Intelligence*, 4(3):256–316, 1992.

Free versions of this paper and all papers referenced in subsequent slides can be found online.

Original context

Matthew Ginsberg created a number of successful systems in applied AI.

Bilattices were introduced to provide a unified framework for a variety of practical inference schemes used in AI, such as **non-monotonic inference**, inference with uncertainty, etc.

Besides purely theoretical interest, they seemed to increase modularity and efficiency of inference implementations.

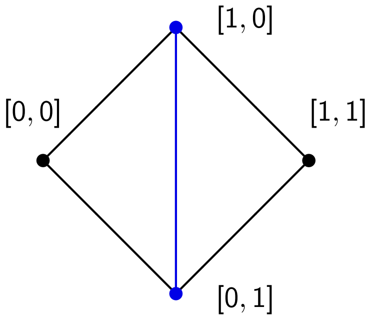
Non-monotonic inference

Plenty of examples of commonsense non-monotonic inference.

E.g. if one knows that Larry is a bird, then one infers that Larry can fly. If one then learns that Larry is actually a pinguin, then one takes this inference back and instead infers that Larry cannot fly. If one then learns that Larry is a magical flying pinguin, then...

Other examples include "negation as failure" in Prolog, etc.

Definition of bilattice



The standard definition of bilattice: 1) $\leq$ and $\sqsubseteq$ form complete lattices; 2) an involutive "weak negation" monotonic with respect to $\sqsubseteq$ and antimonotonic with respect to $\leq$ preserving the appropriate lattice structures (in our case, a reflection with respect to the blue line).

Maximal elements are incompatible with group properties

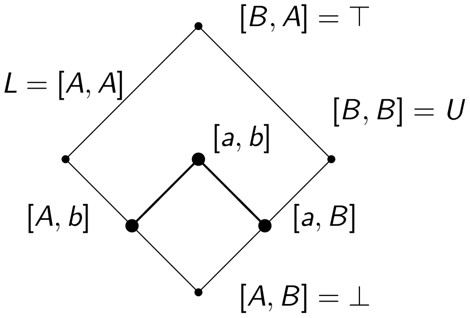
This is not a new situation. For example, a lot of applied math is based on linear algebra, but is implemented with computer representations of real numbers, and these computer representations do not form a group, but one is able to mostly ignore this.

In our case, if we take $A = -\infty, B = \infty$, but omit the segments with infinities, we get a group.

Here we have dropped the requirement that $\leq$ and $\sqsubseteq$ form complete lattices.

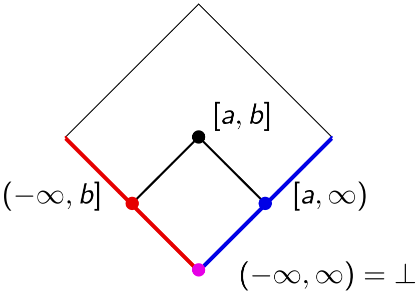
However, the positive and negative subspaces will disappear, and we need them. So let's adjoin their elements, $(-\infty, x]$ and $[x, \infty)$, externally for finite x, and also $\perp = (-\infty, \infty)$.

Decomposition into negative and positive subspaces



$[A, b] = [a, b] \wedge \bot = [a, b] \sqcap L$
$[a, B] = [a, b] \vee \bot = [a, b] \sqcap U$
$[a, b] = [A, b] \sqcup [a, B]$

Bilattice pattern



$(-\infty, b] = [a, b] \wedge \perp$
$[a, \infty) = [a, b] \vee \perp$
$[a, b] = (-\infty, b] \sqcup [a, \infty)$

d-frames

Take two frames L_+ and L_- (the informal intent is for their elements correspond to open sets where the predicates are true and where they are false).

$L = L_+ \times L_-$ is a bilattice.

Introduce $Con, Tot \subseteq L$ with the informal intent that for pairs of open sets $U = \langle U_+, U_- \rangle$, $U \in Con$ when $U_+ \cap U_- = \emptyset$, and $U \in Tot$ when $U_+ \cup U_-$ covers the whole space.

This allows to handle partial inconsistency and the bilattice pattern does appear. (L_+, L_-, Con, Tot) is called a d-frame.

Hahn-Jordan decomposition

Every signed measure μ has unique decomposition into the difference of positive measures μ^+ and μ^-, such that there is no set A with both $\mu^+(A)$ and $\mu^-(A)$ being non-zero: $\mu = \mu^+ - \mu^-$.

The total variation norm, $\|\mu\| = \sup_A \mu^+(A) + \sup_B \mu^-(B)$, makes the space of measures with bounded $\|\mu\|$ a Banach space.

Partial order on the space of signed measures

One takes positive measures as the positive cone in this space.

Our partial order: $\nu < \mu$ iff $\mu - \nu$ is a positive measure.

This is a vector lattice (a Riesz space) and a Banach space, so people call this a Banach lattice.

Bilattice pattern on the space of signed measures

$\mu^+ = \mu \vee 0$ (0 is the zero measure)
$-\mu^- = \mu \wedge 0$

There are two ways to think about $\mu = \mu^+ - \mu^-$.

We can just say that $\mu = \mu^+ + (-\mu^-)$ looks sufficiently similar to our earlier formulas to constitute a bilattice pattern.

Or we can define $\nu \sqsubseteq \mu$ if $\nu^+ \leq \mu^+$ and $\nu^- \leq \mu^-$, and then $\mu = \mu^+ \sqcup (-\mu^-)$, and then it is obviously a bilattice pattern.

Electronic coordinates

These slides are available from my new page on partial inconsistency and vector semantics of programming languages, where **bilattice pattern** plays a prominent role:

`http://www.cs.brandeis.edu/~bukatin/partial_inconsistency.html`

E-mail:

`bukatin@cs.brandeis.edu`