On the Nature of Correspondence between Partial Metrics and Fuzzy Equalities.

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1 Introduction

The correspondence between partial metrics and fuzzy equalities was discovered in 2006 [1]. It was immediately apparent that there was a duality between metric and logical viewpoints, and so the question about the nature of correspondence between partial metrics and fuzzy equalities arose.

Initially, the authors of [1] suggested that we should talk about equivalence between partial metrics and fuzzy equalities up to the choice of dual notation. This suggestion was based on the notion that the duality between metric and logical viewpoints belonged to the metalevel and was a part of the mindset of the practitioners in the respective fields, but did not affect the mathematical structures involved. We refer to this suggestion as the **equivalence approach**.

The equivalence approach remains a legitimate way of viewing this correspondence. In particular, while there is a varierty of possible choices of allowed spaces and morphisms, in all cases studied so far there are (covariant) isomorphisms of the corresponding categories of partial metric spaces and spaces equipped with fuzzy equalities. The induced specialization orders on a partial metric space and the corresponding space equipped with a fuzzy equality also coincide. So, in this sense there seems to be no duality between partial metrics and fuzzy equalities themselves.

Later Mustafa Demirci suggested that the duality between metric and logical viewpoints should nevertheless be brought into formalization of this correspondence by explictly requiring that logical values and distances were respresented by dual structures. We refer to this suggestion as the **duality approach**.

It turns out that the duality approach to understanding this situation is preferrable. It allows to formally express a larger chunk of existing informal mathematical practice, and it allows to do so without explicitly considering the metalevel. Even more importantly, being closer to the respective intuitions of the practitioners in the related fields the duality approach makes it easier to develop applications.

Another aspect of the duality between logical values and distances is that the multiplicative notation is used on the logical side and the additive notation is used on the metric side. This suggests that it might be possible to bring some kind of **exponentiation** into play as well, potentially resulting in a more complicated correspondence and, perhaps, a genuine duality between partial metrics and fuzzy equalities. To the best of our knowledge, this has not been done so far and should be considered an open problem. (It should be noted here that it is not uncommon to start with a metric $d(x,y)$, to express the degree of similarity of x and y as $f(x,y) = e^{-d(x,y)}$, and to call the resulting $f(x,y)$ a fuzzy metric with the appropriate transformation of the axioms of a metric.)

2 Definitions

We provide informal sketches of definitions of quantale-valued partial metrics [3] and quantale-valued sets (sets equipped with quantale-valued fuzzy equalities) [2].

2.1 Quantale-valued Partial Metrics

The quantale V is a complete lattice with an associative and commutative operation $+$, distributed with respect to the arbitrary infima. The unit element is the bottom element 0. The right adjoint to the map $b \mapsto a+b$ is defined as the map $b \mapsto b \dot{-} a = \bigwedge\{c \in V | a+c \geq b\}$. Certain additional conditions are imposed.

Definition 1. *A V-partialpseudometric space is a set X equipped with a map $p : X \times X \to V$ (***partial pseudometric***) subject to the axioms*

- $p(x,x) \leq p(x,y)$
- $p(x,y) = p(y,x)$
- $p(x,z) \leq p(x,y) + (p(y,z) \dot{-} p(y,y))$

2.2 Quantale-valued Sets

The quantale M is a complete lattice with an associative and commutative operation $*$, distributed with respect to the arbitrary suprema. The unit element is the top element 1. The right adjoint to the map $b \mapsto a * b$ is defined as the map $b \mapsto a \Rightarrow b = \bigvee\{c \in V | a * c \sqsubseteq b\}$. Certain additional conditions are imposed.

Definition 2. *An M-valued set is a set X equipped with a map $E : X \times X \to M$ (***fuzzy equality***) subject to the axioms*

- $E(x,y) \sqsubseteq E(x,x)$
- $E(x,y) = E(y,x)$
- $E(x,y) * (E(y,y) \Rightarrow E(y,z)) \sqsubseteq E(x,z)$

3 Equivalence approach

Whenever we have a quantale in the sense of section 2.1, we can equip it with a dual order, $\sqsubseteq = \geq$, and it becomes a quantale in the sense of section 2.2 (and vice versa in the opposite direction).

Define $*$ as $+$, $a \Rightarrow b$ as $b \dot{-} a$, 1 as 0 (and vice versa in the opposite direction).

Then partial pseudometrics and fuzzy equalities coincide as sets of functions. This justifies the equivalence approach.

4 Duality approach

However we found it convenient to press the duality approach as far as possible.

4.1 Partial Ultrametrics Valued in Browerian algebras

For example, consider Ω-sets valued in Heyting algebras. Following the duality approach, on the metric side of things we will talk about partial ultrametrics valued in dual Heyting algebras, but really pressing this approach as far as possible, we'll use the terminology "partial ultrametrics valued in Browerian algebras", and when Ω is actually the algebra of open sets of a topological space X, we will consider partial ultrametrics valued in the algebra of closed sets of the same space.

This helps to understand and establish the following result.

4.2 Sheaves of Sets as Co-sheaves of α-ultrametrics and Non-expansive Maps

Consider a complete Heyting algebra Ω. Consider a corresponding complete Browerian algebra α.

Then every separated pre-sheaf of sets over Ω can be thought of as a separated co-pre-sheaf of α-ultrametrics and non-expansive maps over α.

To develop the necessary intuition one should first consider the case when Ω and α are the algebras of, respectively, open and closed sets of a given topological space.

4.3 Partial Metrics into Non-negative Reals

In the logical situations (arising in domains for denotational semantics, and, in general, in connection with the specialization order on the space of distances) we typically have to flip the ray of non-negative reals, making 0 the top element.

If we press the duality approach as far as possible, the logical counter-part of the partial metrics into non-negative reals ought to be **fuzzy equalities valued in non-positive reals**. So instead of flipping the ray of non-negative reals we replace it with the symmetric ray of non-positive reals.

Partial ultrametrics correspond to idempotent logic (usually, to the ordinary intuitionistic logic). Partial metrics should typically correspond to linear logic, and we think about linear logic as the resource-sensitive logic. So, from the linear logic point of view, it is natural to think about the weight (self-distance) of an element as the work which still needs to be done to make it fully defined. This is the work to be done, something owed, hence negative.

4.4 Intuition Related to Relaxed Metrics

Relaxed metrics typically map (x,y) into an interval number $[l(x,y),u(x,y)]$, where u is usually a partial metric, and l is usually a symmetric function, such that $l(x,y) \leq u(x,y)$.

Function u yields an upper bound for the inequality between “true, underlying x and y”; essentially, “x and y differ no more that $u(x,y)$”, while l yields a lower bound for that, essentially, “x and y differ at least by $l(x,y)$”. There is an intimate relationship between l and negative information, and also between l and tolerances.

From the earlier logical considerations of relaxed metrics we know that u dualizes, but l does not. This means that on the logical side, U becames negative (non-positive, actually), but L remains non-negative.

So, while U represents a work still owed (a work to estimate distance better, actually), and hence negative, L represents a work done, and hence positive (on the logical side). Interestingly enough, the condition $l(x,y) \leq u(x,y)$ on the metric side becomes $L(x,y) + U(x,y) \leq 0$ on the logical side.

If the distance between elements, x and y, is precesely defined (often the case for maximal elements x and y), then $l(x,y) = u(x,y)$, or equivalently $L(x,y) + U(x,y) = 0$, expressing the fact that no further computations are owed.

In general the amount which expresses debt here is not $U(x,y)$, but $L(x,y) + U(x,y) = l(x,y) - u(x,y)$. (Note that $l(x,x)$ is always 0, so the self-distance is always fully owed.)

5 Conclusion

The correspondence between partial metrics and fuzzy equalities allows for the transfer of results and methods between these field, and helps in considering non-trivial interplay between metric and logical situations.

There is a long list of situations where this correspondence should be useful. We only name a few of them here.

It is particularly important to study metric counterparts of the logical research generalizing the fuzzy equalities to the non-commutative case and to categories, in particular results for sets valued in non-commutative quantales (Höhle and Kubiak) and results for sets valued in Grothendieck topologies (Higgs).

Weighted quasi-metrics are a remarkably effective instrument on the metric side, and their logical counterparts would probably be as useful as the global quantale-valued sets which are the logical counterparts of weighted metrics.

References

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